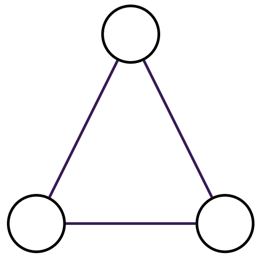
COLOURING GRAPHS

UCL Computer Science Summer School 2026

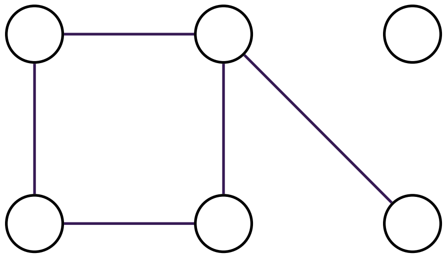
Mathematicians and computer scientists love to use graphs to represent all kinds of information.

A **graph** consists of a collection of **vertices** and **edges** connecting pairs of vertices. They are drawn as a set of circles connected by lines.

exmp. with 3 vertices and 3 edges

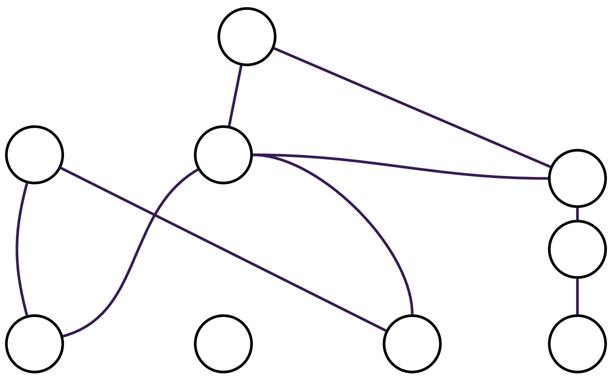


exmp. with 6 vertices and 5 edges



The position of the vertices doesn't matter. Edges can be drawn as curved lines. What matters is which vertices are connected.

exmp. with _ vertices and _ edges

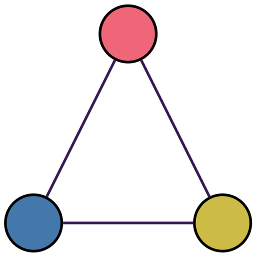


Note: This zine has many interesting problems for you, but the solutions are not given (just like when doing real research).

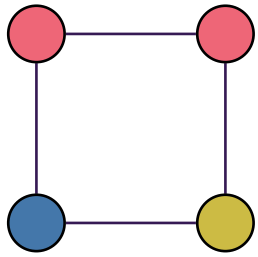
Colourings

A **colouring** of a graph gives every vertex a colour with the constraint that **no edge connects two vertices of the same colour**.

good colouring

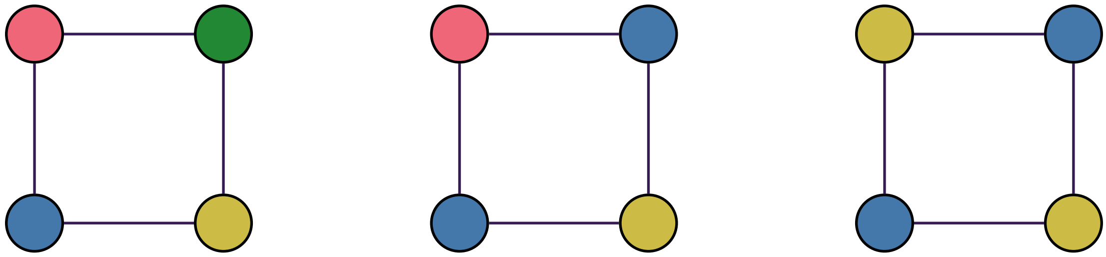


bad colouring



Many colourings exist for one graph. We are looking to use as **few** colours as possible.

the square coloured with 4, 3, and 2 colours

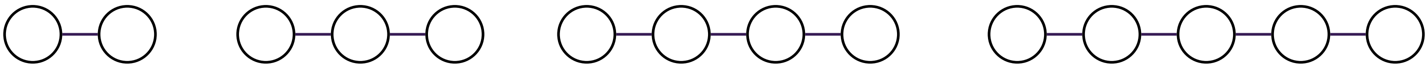


The **chromatic number** of a graph is the minimum number of colours required in any of its colourings.

Paths

The following graphs are called **paths**.

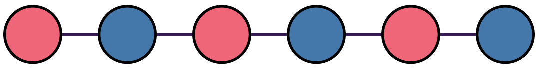
P_2 P_3 P_4 P_5



Every path has chromatic number 2.

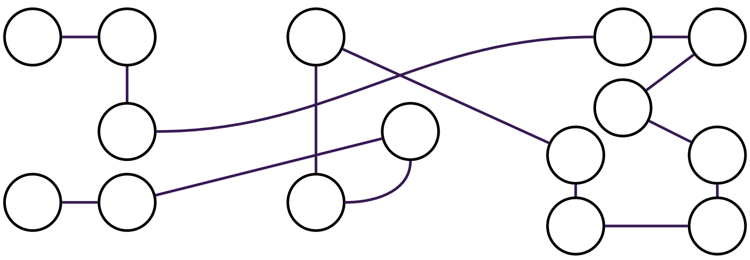
Start at one end and alternate between two colours each time you cross an edge.

P_6 **coloured by alternating red and blue**



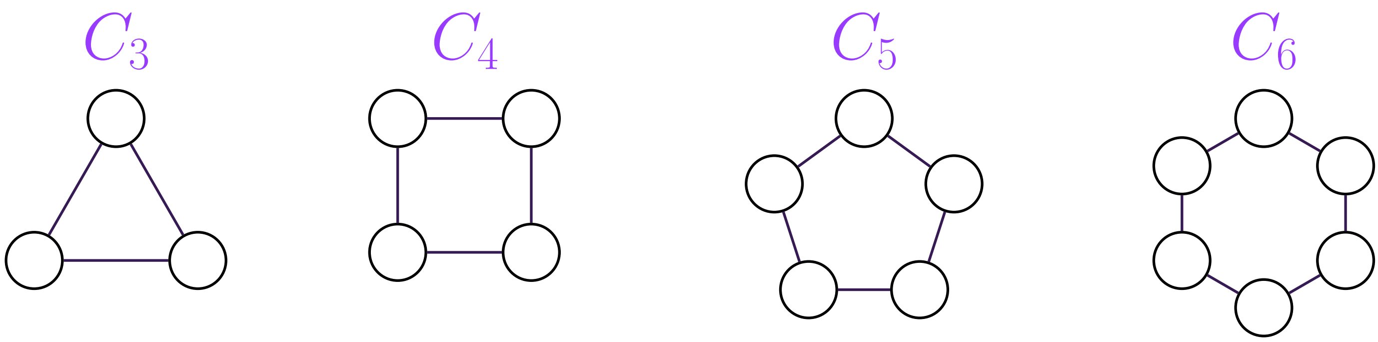
We can't use only one colour because any edge connects two vertices that must be of different colours.

colour this path using two colours



Cycles

The following graphs are called **cycles**.



What is the chromatic number of C_n?

Try to colour the cycles above. The pattern is:

Cycle	C_3	C_4	C_5	C_6
Colours needed	__	__	__	__

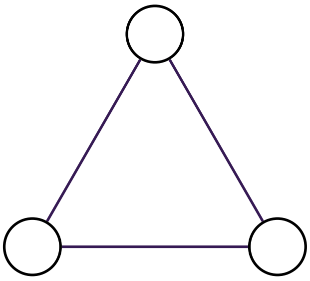
Show that the pattern continues indefinitely.

Hint: Removing a single edge from a cycle results in a path. We know how to colour paths with two colours by alternating. The start and end of the path are connected by the edge we removed. Are the start and end given the same colour?

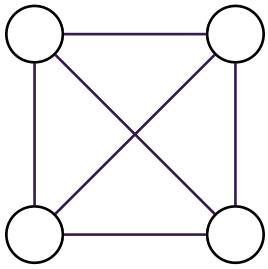
Cliques

The following graphs are called **cliques**.

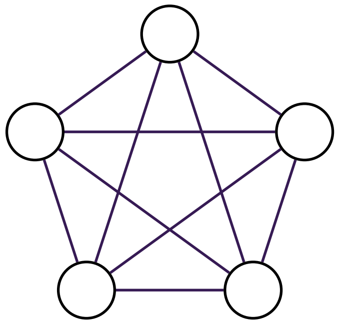
K_3



K_4



K_5

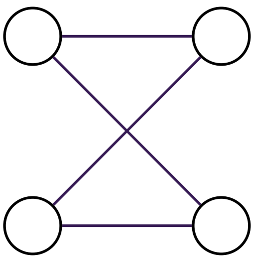


Every vertex is connected to every other vertex, so two vertices never share a colour.

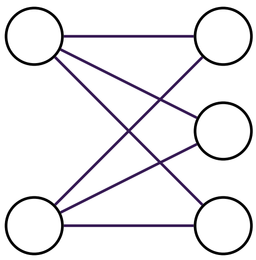
What is the chromatic number of K_n?

The following graphs are called **bicliques**.

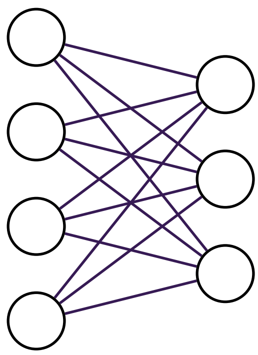
$K_{2,2}$



$K_{2,3}$



$K_{4,3}$

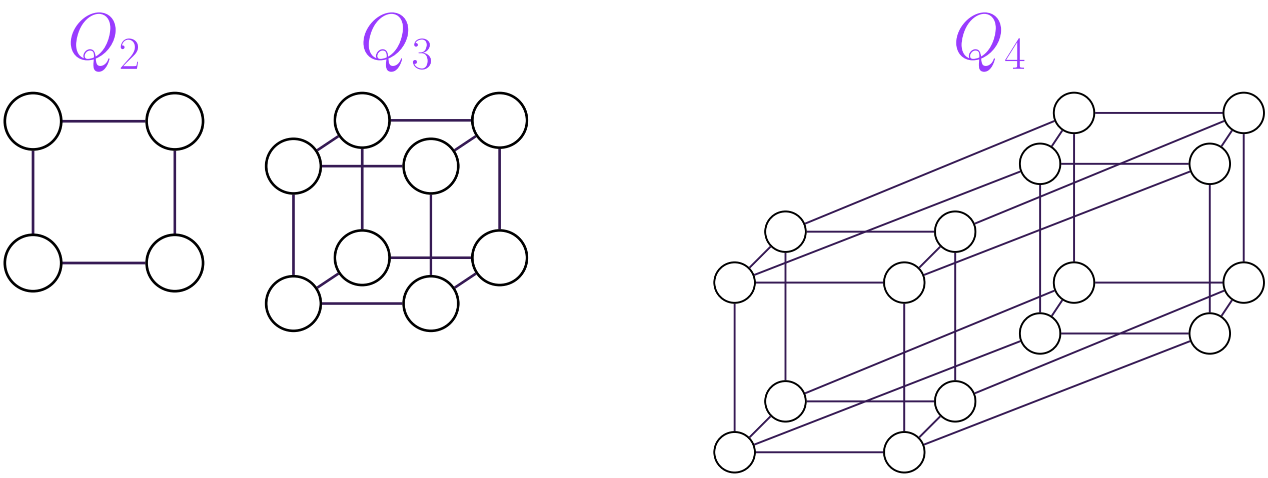


Every vertex on one side is connected to every vertex on the other side.

What is the chromatic number of $K_{n,m}$?

Hypercubes

The following graphs are called **hypercubes**.



The hypercube Q_n is constructed with two copies of Q_{n-1} where corresponding vertices are connected. For example, a cube is two squares where the top left corners are connected, the top right corners are connected, and so on.

What is the chromatic number of Q_n?

Extra: Finding the chromatic number of a graph can be very difficult even for computers. The fastest supercomputer of 2026 could take more than 100 years to solve this problem for graphs with only 100 vertices. If you can find an efficient algorithm to solve this problem, you will have solved one of the biggest conjecture of computer science: the P vs. NP problem.

Graph theory

Graph theory is a subfield of mathematics and computer science that studies graphs and their properties. For example:

- **Clique number:** Size of the largest set of vertices that are all connected to each other in the graph.
- **Shortest paths:** Finding paths between vertices with the least amount of edges.
- **Hamiltonian cycles:** Can you visit every vertex exactly once and return to the start?
- **Vertex cover:** What is the smallest set of vertices that touches every edge?
- **Planarity:** Can the graph be drawn without any edges crossing?

It is popular in maths & CS because of its many applications in research and industry. For example:

- Web mapping platforms use graphs when finding the best itineraries.
- Graphs are widely used in sociology to model people and their interactions.
- DNA sequencing uses graph algorithms to assemble small chunks of DNA into longer sequences.

Here is one last challenge for you:

Draw a graph with 8 vertices, chromatic number 6, and clique number 5.