Universal Quantitative Algebra

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Outline

Definitions

A **signature** Σ is a set of operation symbols, each with an arity, we write $\mathsf{op} : n \in \Sigma$ for an operation of arity $n \in \mathbb{N}$ belonging to Σ.

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Examples

- A **(join)-semilattice** is a set S equipped with an associative, commutative and idempotent binary operation $\oplus : S \times S \to S$. It is a $\Sigma_{\mathcal{P}}$-algebra where $\Sigma_{\mathcal{P}} = \{\oplus : 2\}$ with some nice properties.

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- A **convex algebra** is a set C equipped with binary operations $+_p : C \times C \to C$ for every $p \in [0, 1]$ that satisfy *skewed* associativity, commutativity, and idempotence. It is a $\Sigma_{\mathcal{D}}$-algebra where $\Sigma_{\mathcal{D}} = \{+_p : 2 \mid p \in [0, 1]\}$ with some nice properties.

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- A **pointed set** is a set X equipped with a constant $x \in X$ that we can identify with a function $x : X^0 \to X$. It is a $\Sigma_{-+\mathbf{1}}$-algebra where $\Sigma_{-+\mathbf{1}} = \{\star : 0\}$.

Universal Algebra

Definition
The set of Σ–terms over a set X is defined inductively:

$$\frac{x \in X}{x \in T_\Sigma X} \qquad \frac{t_1 \in T_\Sigma X \qquad \cdots \qquad t_n \in T_\Sigma X}{\mathsf{op}(t_1, \ldots, t_n) \in T_\Sigma X}$$

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The interpretation of operations in an algebra A lifts to terms of $T_\Sigma A$:

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Question

Let $s, t \in T_\Sigma A$, if we know $[\![s]\!] = [\![t]\!]$, what else can we derive?

Birkhoff's Equational Logic

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Theorem
Equational logic is sound and complete.

Monads

Theorem

There is a correspondence between algebraic theories (a signature Σ and axioms E closed under equational logic) and finitary monads $T_{\Sigma,E} : \mathbf{Set} \to \mathbf{Set}$ such that Σ-algebras satisfying E correspond to $T_{\Sigma,E}$-algebras.

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Monad Presentations

Conclusion

We switch from the category **Set** to **Met**.

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A **quantitative Σ–algebra** is a metric space (A, d) and a Σ-algebra on the same carrier, i.e. interpretations $[\![\mathsf{op}]\!] : A^n \to A$ for every $\mathsf{op} : n \in \Sigma$.

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Example

Let $\Sigma_{\mathcal{D}} = \{+_p : 2\}_{p\in(0,1)}$ and (A, d) be a metric space. We denote by $(\mathcal{D}A, \widehat{d})$ the space of finite probability distributions on A with the Kantorovich metric.

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$$[\![+_p]\!] : \mathcal{D}A \times \mathcal{D}A \to \mathcal{D}A = (\varphi, \psi) \mapsto p\varphi + (1-p)\psi$$

yield a quantitative $\Sigma_{\mathcal{D}}$–algbera.

We can now work with more information on terms: equality and distance. Thus we ask:

Question

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The information on distance is now also relevant on variables, e.g.:

Question

If we know $d(\llbracket s \rrbracket, \llbracket t \rrbracket) \leq \varepsilon$ but only if the variables x and y are at distance δ, what else can we derive?

Quantitative Equational Logic

We introduce a binary predicate $=_\varepsilon$ which we interpret as the two inputs having distance *at most* ε, and the context (variables) is now a metric space. We add:

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Theorem

Quantitative equational logic is sound and complete.

Hausdorff Lifting

The Hausdorff lifting of the powerset monad $\widehat{\mathcal{P}}_{\mathsf{H}}$ is defined by

$$(X, d) \mapsto (\mathcal{P}X, d_{\mathsf{H}}) \text{ where } d_{\mathsf{H}}(S, T) = \max\left\{\max_{x\in S}\min_{y\in T} d(x, y), \max_{y\in T}\min_{x\in S} d(x, y)\right\}.$$

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The unit and multiplication are the same as those of $\mathcal{P}$. Algebras for $\widehat{\mathcal{P}}_{\mathsf{H}}$ correspond to quantitative $\Sigma_{\mathcal{P}}$-algebras satisfying:

$$\begin{aligned}
x &\vdash x \oplus x = x && \text{(idempotent)} \\
x, y &\vdash x \oplus y = y \oplus x && \text{(commutative)} \\
x, y, z &\vdash x \oplus (y \oplus z) = (x \oplus y) \oplus z && \text{(associative)} \\
x =_{\varepsilon} x', y =_{\varepsilon'} y' &\vdash x \oplus y =_{\max\{\varepsilon, \varepsilon'\}} x' \oplus y' && \text{(Hausdorff)}
\end{aligned}$$

Hausdorff Lifting

The Hausdorff lifting of the powerset monad $\widehat{\mathcal{P}}_{\mathsf{H}}$ is defined by

$$(X,d) \mapsto (\mathcal{P}X, d_{\mathsf{H}}) \text{ where } d_{\mathsf{H}}(S,T) = \max\left\{\max_{x\in S}\min_{y\in T} d(x,y), \max_{y\in T}\min_{x\in S} d(x,y)\right\}.$$

The unit and multiplication are the same as those of $\mathcal{P}$. Algebras for $\widehat{\mathcal{P}}_{\mathsf{H}}$ correspond to quantitative $\Sigma_{\mathcal{P}}$-algebras satisfying:

$$x \vdash x \oplus x = x \qquad \text{(idempotent)}$$

$$x, y \vdash x \oplus y = y \oplus x \qquad \text{(commutative)}$$

$$x, y, z \vdash x \oplus (y \oplus z) = (x \oplus y) \oplus z \qquad \text{(associative)}$$

$$x =_{\varepsilon} x', y =_{\varepsilon'} y' \vdash x \oplus y =_{\max\{\varepsilon,\varepsilon'\}} x' \oplus y' \qquad \text{(Hausdorff)}$$

In all these algebras, $\oplus$ is a nonexpansive operation $(X,d)^2 \to (X,d)$.

Not Hausdorff Lifting

After removing that last quantitative equation, we get algebras for the monad $\widehat{\mathcal{P}}$ defined by:

$$(X, d) \mapsto (\mathcal{P}X, \widehat{d}) \text{ where } \widehat{d}(S, T) = \begin{cases} 0 & S = T \\ d(x, y) & S = \{x\} \text{ and } T = \{y\} \\ 1 & \text{otherwise} \end{cases}.$$

ŁK Lifting

The Łukaszyk–Karmowski lifting of the distribution monad $\widehat{\mathcal{D}}_{\text{ŁK}}$ is defined by:

$$(X, d) \mapsto (\mathcal{D}X, d_{\text{ŁK}}) \text{ where } d_{\text{ŁK}}(\varphi, \psi) = \sum_{x,x' \in X} \varphi(x)\psi(x')d(x, x').$$

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$$\begin{aligned}
x &\vdash x +_p x = x && \text{(idempotent)} \\
x, y &\vdash x +_p y = y +_{1-p} x && \text{(skew comm.)} \\
x, y, z &\vdash (x +_q y) +_p z = x +_{pq} (y +_{\frac{p(1-q)}{1-pq}} z) && \text{(skew assoc.)} \\
x =_{\varepsilon_1} y, x =_{\varepsilon_2} z &\vdash x =_{p\varepsilon_1 + (1-p)\varepsilon_2} y +_p z && \text{(ŁK)}
\end{aligned}$$

Outline

Universal Algebra

Universal Quantitative Algebra

Monad Presentations

Conclusion

Set Presentations

Given an algebraic theory (Σ, E), the free algebra monad $T_{\Sigma,E}$ is given by

$$X \mapsto T_\Sigma X/{\equiv_E}, \text{ where } {\equiv_E} = \{(s,t) \mid X \vdash s = t \in E\}.$$

A Σ-algebra satisfying E is the same thing as a $T_{\Sigma,E}$-algebra.

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Definition

A monad M on **Set** is **presented** by (Σ, E) if there is a monad isomorphism $\rho : T_{\Sigma,E} \cong M$.

Met Presentations

Given a quantitative algebraic theory $(\Sigma, \widehat{E})$, the free quantitative algebra monad $\widehat{T}_{\Sigma,\widehat{E}}$ is given by

$$\mathbf{X} \mapsto (T_\Sigma X/{\equiv_{\widehat{E}}}, d_{\widehat{E}}), \text{ where } \begin{array}{r} \equiv_{\widehat{E}} = \{(s,t) \mid \mathbf{X} \vdash s = t\} \\ d_{\widehat{E}}([s],[t]) = \inf\{\varepsilon \in [0,\infty] \mid \mathbf{X} \vdash s =_\varepsilon t\} \end{array}.$$

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Definition

A monad $\widehat{M}$ on **Met** is **presented** by $(\Sigma, \widehat{E})$ if there is a monad isomorphism $\widehat{\rho} : \widehat{T}_{\Sigma,\widehat{E}} \cong \widehat{M}$.

Let (M, η, μ) be a monad on **Set**, and (Σ, E) be an algebraic presentation for it via $\rho : T_{\Sigma,E} \cong M$.

Definitions

A **monad lifting** of M to **Met** is a monad $\widehat{M} : \mathbf{Met} \to \mathbf{Met}$ whose functor, unit and multiplication coincide with those of M after applying $U : \mathbf{Met} \to \mathbf{Set}$.

Lifting Presentations

Let (M, η, μ) be a monad on **Set**, and (Σ, E) be an algebraic presentation for it via $\rho : T_{\Sigma,E} \cong M$.

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A **quantitative extension** of E is a quantitative algebraic theory $\widehat{E}$ on the same signature Σ satisfying for all $\mathbf{X} \in \mathbf{Met}$ and $s, t \in T_\Sigma X$,

$$X \vdash s = t \in E \iff \mathbf{X} \vdash s = t \in \widehat{E}.$$

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Theorem

There is a correspondence between monad liftings of M and quantitative extensions of E.

Extension to Lifting (Easy)

- The equivalence
$$X \vdash s = t \in E \iff \mathbf{X} \vdash s = t \in \widehat{E}$$
really says that $\equiv_E = \equiv_{\widehat{E}}$, so the functors $T_{\Sigma,E}$ and $\widehat{T}_{\Sigma,\widehat{E}}$ agree on sets.

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- It follows from the syntactic definitions that the units and multiplications also coincide, hence $\widehat{T}_{\Sigma,\widehat{E}}$ is a monad lifting of $T_{\Sigma,E}$.
- Via the isomorphism $\rho : T_{\Sigma,E} \cong M$, we can construct the monad lifting by
$$\widehat{M}(X,d) = (MX, \widehat{d}), \text{ where } \widehat{d}(m,m') = d_{\widehat{E}}(\rho^{-1}m, \rho^{-1}m').$$

Lifting to Extension

- Put some equations in $\widehat{E}$:

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- Put some quantitative equations in $\widehat{E}$:

$$\text{For all } (X,d) \in \mathbf{Met} \text{ and } s,t \in T_\Sigma X, \text{ add } (X,d) \vdash s =_{\widehat{d}(\rho[s],\rho[t])} t \text{ to } \widehat{E}.$$

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$$\text{For all } (X, d) \in \mathbf{Met} \text{ and } s, t \in T_\Sigma X, \text{ add } (X, d) \vdash s =_{\widehat{d}(\rho[s], \rho[t])} t \text{ to } \widehat{E}.$$

- Show that nothing else is entailed by exhibiting $\widehat{M}(\mathbf{X})$ as the free Σ-algebra satisfying $\widehat{E}$ generated by $\mathbf{X}$.

Outline

To Be Done

- Make the result more categorical in flavor.
- What about infinitary theories?
- What about composing monads?
- Further simplify the entry point to quantitative algebraic reasoning.

Merci !