Lifting Algebraic Reasoning to Generalized Metric Spaces

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Outline

Let's Talk About Hummus

```
into a food processor
add 300g strained cooked chickpeas
add 3 cloves of garlic
add 75g tahini
add 25mL lemon juice
blend for 5 minutes
season to taste
```

```
into a food processor
add 300g strained cooked chickpeas
add 4 cloves of garlic
add 75g tahini
add 25mL lemon juice
blend for 5 minutes
add 5 ice cubes
blend for 3 minutes
season to taste
```

Two different recipes but the result is hummus. We can compare the recipes:

- Does recipe 2 taste better than recipe 1?
- How different do the results taste?
- Which recipe takes longer?

Example ([Neu51])

```
return fairCoin(H,T)
```

```
do
    x = biasedCoin(H,T)
    y = biasedCoin(H,T)
while (x == y)
return x
```

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do
    x = biasedCoin(H,T)
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```

As long as the bias is consistent and not total ($0\% < p < 100\%$), the two programs have the same behavior.

Example (Guaranteed Termination)

```
return fairCoin(H,T)
```

```
i = 0
do
    i = i + 1
    x = biasedCoin(H,T)
    y = biasedCoin(H,T)
while (x == y) AND i <= 1000
return x
```

The second program is very close to being a fair coin flip.

Algebraic Semantics

We can see programming language syntax as operations on the set $\mathfrak{P}$ of programs.

- Composition of two lines done with a semicolon:

$$; : \mathfrak{P} \times \mathfrak{P} \to \mathfrak{P} \text{ sends } (C_1, C_2) \text{ to } C_1; C_2.$$

- Random branching:

$$\texttt{fairCoin} : \mathfrak{P}^2 \to \mathfrak{P} \text{ sends } (C_1, C_2) \text{ to } \texttt{fairCoin}(C_1, C_2).$$

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Equivalences of programs can be proven with algebraic reasoning.
With the axioms $P; (Q; R) = (P; Q); R$ and $\texttt{fairCoin}(P, P) = P$, we can show

$$\texttt{fairCoin}(P; (Q; R),\ (P; Q); R) = P; (Q; R).$$

Convex Algebras

Let us focus on one signature $\Sigma = \{+_p : 2 \mid p \in (0,1)\}$.

- Starting with a set of atomic instructions/states $X = \{x, y, z, w, \cdots\}$, the programs we can write are called Σ-terms, e.g.

$$x +_p y \qquad x \qquad (x +_p y) +_q (w +_p z) \qquad (((w +_q w) +_p z) +_q x).$$

- Understanding $+_p$ as a probabilistic choice (c.f. `fairCoin` and `biasedCoin`), we postulate the axioms of convex algebras:

$$x +_p x = x \qquad x +_p y = y +_{1-p} x \qquad (x +_q y) +_p z = x +_{pq} (y +_{\frac{p(1-q)}{1-pq}} z).$$

- Two terms are equivalent if and only if they represent the same probability distribution. We can reason algebraically about probability distributions!

$$\texttt{fairCoin}(\texttt{H}, \texttt{T}) = \texttt{fairCoin}(\texttt{fairCoin}(\texttt{H}, \texttt{T}), \texttt{fairCoin}(\texttt{H}, \texttt{T}))$$
$$H +_{0.5} T = (H +_{0.5} T) +_{0.5} (H +_{0.5} T)$$

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Question

Can we reason algebraically about distances between distributions?

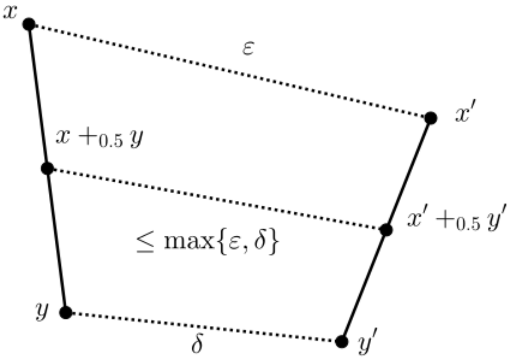
Mardare, Panangaden, and Plotkin's Answer

Working in **Met**: objects are metric spaces $(X, d_\mathbf{X} : X \times X \to [0, \infty])$, and morphisms are nonexpansive functions: $f : X \to Y$ such that

$$d_\mathbf{Y}(f(x), f(x')) \leq d_\mathbf{X}(x, x').$$

A convex algebra in **Met** is a metric space $(A, d_\mathbf{A})$ with nonexpansive operations

$$+_p : (A, d_\mathbf{A}) \times (A, d_\mathbf{A}) \to (A, d_\mathbf{A}).$$



Mardare, Panangaden, and Plotkin's Answer

Example

For any space (X, d), there is the Kantorovich metric on distributions $(\mathcal{D}X, d_{\mathrm{K}})$.
Convex combinations are nonexpansive operations $(\mathcal{D}X, d_{\mathrm{K}})^2 \to (\mathcal{D}X, d_{\mathrm{K}})$.

$$(\varphi +_p \psi)(x) = p\varphi(x) + (1-p)\psi(x).$$

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In addition to the previous equations, this convex algebra satisfies an implication:

$$\big(d_{\mathrm{K}}(\varphi, \varphi') \le \varepsilon \text{ and } d_{\mathrm{K}}(\psi, \psi') \le \delta\big) \implies d_{\mathrm{K}}(\varphi +_p \psi, \varphi' +_p \psi') \le p\varepsilon + (1-p)\delta.$$

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In [MPP16]:

- Replace $d(x, y) \leq \varepsilon$ with $x =_\varepsilon y$ and build an implicational logic.

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In [MPP16]:

- Replace $d(x, y) \leq \varepsilon$ with $x =_\varepsilon y$ and build an implicational logic.
- Construct free algebras with $(\mathcal{D}X, +_p, d_K)$ as an example.
- Axiomatization of metrics, e.g. Kantorovich, Hausdorff, total variation.

In following papers, more results generalized from universal algebra: HSP theorems, composite theories, monad-theory correspondences, more axiomatizations, etc.

Outline

Our Contributions

- Replace metric spaces with generalized metric spaces, this includes pseudo-, quasi-, ultra-metric spaces (already in [MSV22]), posets, simple graphs, probabilistic metric spaces, etc. (c.f. [FMS21]).

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- Replace metric spaces with generalized metric spaces, this includes pseudo-, quasi-, ultra-metric spaces (already in [MSV22]), posets, simple graphs, probabilistic metric spaces, etc. (c.f. [FMS21]).
- **Allow operations to be arbitrary functions** (also in [MSV22]).
  - Motivation in [Bac+18a; Bac+18b; Cas+21; DL+22].

$$+_p : (\mathcal{D}X, d_{\text{ŁK}}) \times (\mathcal{D}X, d_{\text{ŁK}}) \to (\mathcal{D}X, d_{\text{ŁK}}) \text{ is not nonexpansive.}$$

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- Provide a sound and complete logic that is not *implicational* (c.f. [FMS21]).
- **Sufficient condition and construction for quantitative algebraic presentations for monads on GMet.**

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$[0, \infty]$-spaces

While metrics are relevant in practice, the essence of [MPP16]'s solution is in the following isomorphism of categories.

Definition ($[0, \infty]$Spa)

A $[0, \infty]$**-space** is a set A equipped with a distance function $d_A : A \times A \to [0, \infty]$. Morphisms are nonexpansive maps: $f : A \to B$ such that $d_B(f(a), f(a')) \leq d_A(a, a')$.

Definition ($[0, \infty]$Str)

A $[0, \infty]$**-structure** is a set A equipped with a family of binary predicates $=_\varepsilon \subseteq A \times A$ indexed by $[0, \infty]$ satisfying

$$\varepsilon \leq \varepsilon' \implies =_\varepsilon \subseteq =_{\varepsilon'} \quad \text{and} \quad =_{\inf S} = (\cap_{\varepsilon \in S} =_\varepsilon).$$

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Proposition (2.21)

$[0,\infty]\mathbf{Spa} \cong [0,\infty]\mathbf{Str}$ *by understanding* $a =_\varepsilon a'$ *as* $d_A(a, a') \leq \varepsilon$.

L-spaces

This remains true for any complete lattice L, e.g. $[0, \infty]$ or $[0, 1]$ or $\{0, 1\}$ (examples are usually quantales).

Definition (**LSpa**, 2.11)

An **L-space** is a set A equipped with a distance function $d_A : A \times A \to \mathsf{L}$.
Morphisms are nonexpansive maps: $f : A \to B$ such that $d_B(f(a), f(a')) \leq d_A(a, a')$.

Definition (**LStr**, 2.19)

An **L-structure** is a set A equipped with a family of binary predicates $=_\varepsilon \subseteq A \times A$ indexed by L satisfying

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Quantitative Algebras

Definition (3.1)

Given a signature $\Sigma = \{\mathsf{op}_i : n_i\}_{i \in I}$, a **quantitative Σ-algebra** is an L-space $(A, d_{\mathbf{A}})$, and an interpretation $[\![\mathsf{op}]\!]_A : A^n \to A$ in **Set** for every $\mathsf{op} : n \in \Sigma$.

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Examples

- The metric space $(\mathcal{D}X, d_{\mathrm{K}})$ with convex combinations $+_p : \mathcal{D}X \times \mathcal{D}X \to \mathcal{D}X$.
- The space $(\mathcal{D}X, d_{\mathrm{ŁK}})$ with convex combinations.
- The real numbers $\mathbb{R}$ with the Euclidean metric d and all the ring operations.

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- The real numbers $\mathbb{R}$ with the Euclidean metric d and all the ring operations.

  ⚠ Addition and multiplication $+, \times : (\mathbb{R}, d) \times (\mathbb{R}, d) \to (\mathbb{R}, d)$ are not in **Met**.

$$\begin{array}{ccccc} 1 & + & 1 & = & 2 \\ {\scriptstyle 1}\vert & & {\scriptstyle 1}\vert & & {\scriptstyle 2}\vert \\ 2 & + & 2 & = & 4 \end{array}$$

Outline

Quantitative Equations

A classical equation is a judgment $X \vdash s = t$, where s and t are Σ-terms over the variables in a set X. An algebra $\mathbb{A}$ satisfies it if for all $\iota : X \to A$, $[\![s]\!]^{\iota}_{A} = [\![t]\!]^{\iota}_{A}$. The meaning of $X \vdash \quad$ is **universal quantification**.

This is not enough for quantitative algebras. How can you assert that the interpretation of $\mathsf{f} : 1$ is a contraction: the distance between $\mathsf{f}x$ and $\mathsf{f}y$ is less than the distance between x and y.

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Definition (3.8)

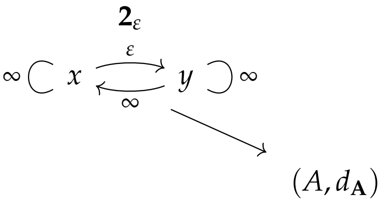
A **quantitative equation** is a judgment

$$(X, d_{\mathbf{X}}) \vdash s = t \quad \text{or} \quad (X, d_{\mathbf{X}}) \vdash s =_\varepsilon t,$$

where $(X, d_{\mathbf{X}})$ is an L-space and s, t are Σ-terms over X.
It is **satisfied** by a quantitative algebra $(A, [\![-]\!]_A, d_{\mathbf{A}})$ if for all **nonexpansive** assignments $\hat{\iota} : (X, d_{\mathbf{X}}) \to (A, d_{\mathbf{A}})$,

$$[\![s]\!]_A^{\hat{\iota}} = [\![t]\!]_A^{\hat{\iota}} \quad \text{or} \quad d_{\mathbf{A}}([\![s]\!]_A^{\hat{\iota}}, [\![t]\!]_A^{\hat{\iota}}) \leq \varepsilon.$$

Examples



Examples ($\Sigma = \{\mathsf{f}:1, +:2\}$)

- If $\mathbf{2}_\varepsilon \vdash \mathsf{f}x =_\varepsilon \mathsf{f}y$ is satisfied $\forall\varepsilon$, then $[\![\mathsf{f}]\!]_A : (A, d_{\mathbf{A}}) \to (A, d_{\mathbf{A}})$ is *nonexpansive*.

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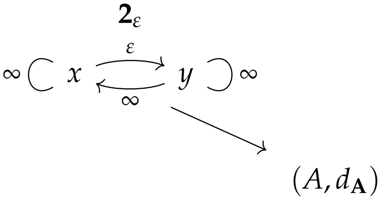

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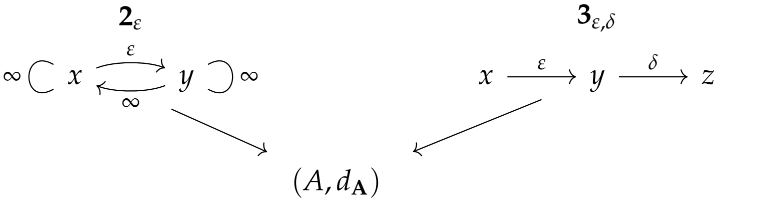

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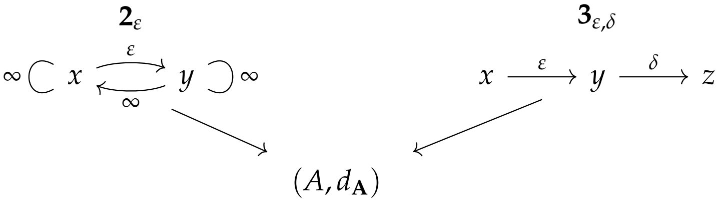

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Following the last example, we define **GMet** to be a full subcategory of **QAlg**($\emptyset$) defined by a collection of quantitative equations, e.g. **Met**, **UMet**, **Poset**, **Grph**, etc.

Some Rules

$$\frac{\mathbf{X} \vdash s = t}{\mathbf{X} \vdash t = s}\ \textsc{Symm} \qquad \frac{\mathsf{op} : n \in \Sigma \qquad \forall 1 \leq i \leq n,\ \mathbf{X} \vdash s_i = t_i}{\mathbf{X} \vdash \mathsf{op}(s_1, \ldots, s_n) = \mathsf{op}(t_1, \ldots, t_n)}\ \textsc{Cong}$$

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$$\frac{}{\mathbf{X} \vdash s =_\top t}\ \textsc{Top} \qquad \frac{d_{\mathbf{X}}(x, x') = \varepsilon}{\mathbf{X} \vdash x =_\varepsilon x'}\ \textsc{Vars} \qquad \frac{\forall i, \mathbf{X} \vdash s =_{\varepsilon_i} t \qquad \varepsilon = \inf_i \varepsilon_i}{\mathbf{X} \vdash s =_\varepsilon t}\ \textsc{Cont}$$

$$\frac{\sigma : Y \to \mathcal{T}_\Sigma X \qquad \mathbf{Y} \vdash s =_\varepsilon t \qquad \forall y, y' \in Y, \mathbf{X} \vdash \sigma(y) =_{d_{\mathbf{Y}}(y,y')} \sigma(y')}{\mathbf{X} \vdash \sigma^*(s) =_\varepsilon \sigma^*(t)}\ \textsc{SubQ}$$

$$\frac{\sigma : Y \to \mathcal{T}_\Sigma X \qquad \{y_i =_{\varepsilon_i} y'_i\} \vdash s =_\varepsilon t}{\{\sigma(y_i) =_{\varepsilon_i} \sigma(y'_i)\} \vdash \sigma^*(s) =_\varepsilon \sigma^*(t)}\ \text{Sub[MPP16]}$$

Some Rules

$$\frac{\mathbf{X} \vdash s = t}{\mathbf{X} \vdash t = s}\ \textsc{Symm} \qquad \frac{\mathsf{op} : n \in \Sigma \qquad \forall 1 \leq i \leq n,\ \mathbf{X} \vdash s_i = t_i}{\mathbf{X} \vdash \mathsf{op}(s_1, \ldots, s_n) = \mathsf{op}(t_1, \ldots, t_n)}\ \textsc{Cong}$$

$$\frac{}{\mathbf{X} \vdash s =_\top t}\ \textsc{Top} \qquad \frac{d_{\mathbf{X}}(x, x') = \varepsilon}{\mathbf{X} \vdash x =_\varepsilon x'}\ \textsc{Vars} \qquad \frac{\forall i, \mathbf{X} \vdash s =_{\varepsilon_i} t \qquad \varepsilon = \inf_i \varepsilon_i}{\mathbf{X} \vdash s =_\varepsilon t}\ \textsc{Cont}$$

$$\frac{\sigma : Y \to \mathcal{T}_\Sigma X \qquad \mathbf{Y} \vdash s =_\varepsilon t \qquad \forall y, y' \in Y, \mathbf{X} \vdash \sigma(y) =_{d_{\mathbf{Y}}(y,y')} \sigma(y')}{\mathbf{X} \vdash \sigma^*(s) =_\varepsilon \sigma^*(t)}\ \textsc{SubQ}$$

Theorem (3.69 & 3.76)

Quantitative equational logic is sound and complete.

Outline

Free Quantitative Algebras

Given a collection $\hat{E}$ of quantitative equations, the quantitative variety $\mathbf{QAlg}(\Sigma, \hat{E})$ of quantitative Σ-algebras satisfying $\hat{E}$ has free algebras over **GMet**, yielding a monad on **GMet**.

$$\mathbf{GMet} \underset{U}{\overset{\widehat{\mathbb{T}}_{\Sigma,\hat{E}}}{\rightleftarrows}} \mathbf{QAlg}(\Sigma, \hat{E}) \quad (\bot)$$

Free Quantitative Algebras

Given a collection $\hat{E}$ of quantitative equations, the quantitative variety $\mathbf{QAlg}(\Sigma, \hat{E})$ of quantitative Σ-algebras satisfying $\hat{E}$ has free algebras over **GMet**, yielding a monad on **GMet**.

$$\mathbf{GMet} \underset{U}{\overset{\widehat{\mathbb{T}}_{\Sigma,\hat{E}}}{\rightleftarrows}} \mathbf{QAlg}(\Sigma, \hat{E}) \quad (\bot)$$

Examples

- The Kantorovich monad $\mathcal{D}_{\mathrm{K}} : \mathbf{Met} \to \mathbf{Met} = (X, d) \mapsto (\mathcal{D}X, d_{\mathrm{K}})$ in [MPP16].
- The ŁK monad $\mathcal{D}_{\text{ŁK}} : \mathbf{DMet} \to \mathbf{DMet} = (X, d) \mapsto (\mathcal{D}X, d_{\text{ŁK}})$ in [MSV22] and 3.102.
- The Hausdorff monad $\mathcal{P}^{\uparrow}_{\mathrm{ne}} : \mathbf{Met} \to \mathbf{Met}$ in [MPP16].
- The 'trivial' powerset monad $\widehat{\mathcal{P}} : \mathbf{Met} \to \mathbf{Met}$ in 3.100

Lifting-Extension Correspondence

Monad-theory correspondences are shown in [FMS21; Ros21; Adá22; ADV23; Ros24] with two caveats: arities can be infinite, and operations are nonexpansive (thus, monads are enriched).

Lifting-Extension Correspondence

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Most examples of quantitative algebraic theories present monad liftings, and they are based on classical algebraic theories.

$$\begin{array}{ccc} \mathbf{GMet} & \xrightarrow{\widehat{M}} & \mathbf{GMet} \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U} \\ \mathbf{Set} & \xrightarrow{M} & \mathbf{Set} \end{array} \qquad \widehat{M}(X,d) = (MX, \widehat{d})$$

Theorem (3.96, 3.98, 3.99)

$$\widehat{M} \textit{ is a monad lifting of a monad } M \textit{ presented by } (\Sigma, E).$$
$$\Updownarrow$$
$$\widehat{M} \textit{ is presented by } (\Sigma, \hat{E}), \textit{ where } \hat{E} \textit{ is an extension of } E.$$

Outline

Future Work

- Can we combine our work with [FMS21] to reason algebraically over relational structures? [JMU24] does this for total operations.
- Is there a functorial semantics framework exactly as expressive as ours? [Ros24] answered positively for Mardare et al.'s original quantitative algebras.
- How to compose two liftings of monads when their underlying **Set** monads compose via composite theories? Examples in [MV20; MSV21].
- Further simplify the entry point to quantitative algebraic reasoning (find lots of examples).
- Quantitative diagrammatic reasoning!

Merci !

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There is a well-known equivalent, more categorical, definition of equation.

Definition (1.50)

An **abstract equation** in $\mathbf{Alg}(\Sigma)$ is a surjective homomorphism $e : \mathcal{T}_\Sigma X \twoheadrightarrow \mathbb{Y}$.
We say that an algebra $\mathbb{A} \in \mathbf{Alg}(\Sigma)$ satisfies e if for any assignment $\iota : X \to A$, the function $[\![-]\!]^\iota_A$ factors through e in $\mathbf{Alg}(\Sigma)$:

$$[\![-]\!]^\iota_A = \mathcal{T}_\Sigma X \xrightarrow{e} \mathbb{Y} \xrightarrow{h} \mathbb{A}.$$

Proposition (1.51 & 1.52)

Equations and abstract equations are equivalent in terms of expressiveness.

Abstract Quantitative Equations

We can generalize to quantitative algebras as follows.

Definition (3.61)

An **abstract quantitative equation** is a surjective nonexpansive homomorphism $e : \widehat{\mathcal{T}_\Sigma}\mathbf{X} \to \hat{\mathbb{Y}}$. We say that a quantitative algebra $\hat{\mathbb{A}}$ satisfies e if for any nonexpansive assignment $\hat{\imath} : \mathbf{X} \to \mathbf{A}$, the homomorphism $\hat{\imath}^\sharp$ factors through e in $\mathbf{QAlg}(\Sigma)$:

$$\hat{\imath}^\sharp = \widehat{\mathcal{T}_\Sigma}\mathbf{X} \xrightarrow{e} \hat{\mathbb{Y}} \xrightarrow{h} \hat{\mathbb{A}}.$$

Proposition (3.62 & 3.63)

Quantitative equations (as we define them) and abstract quantitative equations are equivalent in terms of expressiveness.

Example

We can't take e to be epimorphisms, because $e : \mathbb{Q} \twoheadrightarrow \mathbb{R}$ is satisfied by $\mathbb{R}$ and not $\mathbb{Q}$.

Easy Half of Variety Theorem

Definition (3.22)

A homomorphism $h : \hat{\mathbb{A}} \to \hat{\mathbb{B}}$ is called **reflexive** if its underlying nonexpansive map $h : \mathbf{A} \to \mathbf{B}$ is a split epimorphism. Equivalently, for any subspace $\mathbf{B}' \subseteq \mathbf{B}$, there is a subspace $\mathbf{A}' \subseteq \mathbf{A}$ such that $h(A') = B'$ and the (co)restriction $h : \mathbf{A}' \to \mathbf{B}'$ is an isomorphism.

c.f. c-reflexive homomorphisms in [MPP17]: the quantification of $\mathbf{B}'$ is restricted to subspaces of cardinality smaller than c. Hence, h is reflexive if and only if it is c-reflexive for all c.

Theorem (3.23)

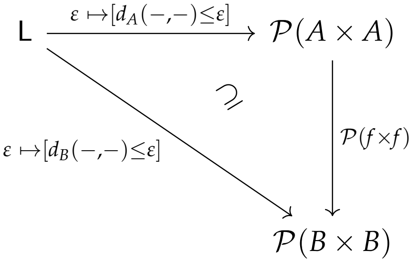
For any class of quantitative equations $\hat{E}$, the category $\mathbf{QAlg}(\Sigma, \hat{E})$ is closed under reflexive homomorphic images, subalgebras, and products.

Theorem (3.65)

A subcategory $\mathbf{K}$ of $\mathsf{L}\mathbf{Spa}$ is closed under subspaces (up to isomorphisms) and products if and only if it is a category $\mathbf{GMet} = \mathbf{QAlg}(\emptyset, \hat{E})$.

Constructing **LSpa**

L**Spa** is a lax comma category of continuous functors $\mathsf{L} \to (\mathcal{P}(A \times A), \subseteq)$:



The lax commutativity of the triangle means for any $\varepsilon \in \mathsf{L}$,

$$\mathcal{P}(f \times f)\{(a,a') \mid d_A(a,a') \leq \varepsilon\} \subseteq \{(b,b') \mid d_B(b,b') \leq \varepsilon\}$$
$$\{(f(a),f(a')) \mid d_A(a,a') \leq \varepsilon\} \subseteq \{(b,b') \mid d_B(b,b') \leq \varepsilon\}$$
$$d_B(f(a),f(a')) \leq d_A(a,a')$$

Compass of Lawvere Theories

- A model of a Lawvere theory $\mathcal{L}_{\Sigma,E}$ valued in **Met** is a quantitative algebra satisfying the classical equations in E with operations that are nonexpansive:

$$F(\mathsf{op} : n \to 1) : \mathbf{A} \times \overset{n}{\cdots} \times \mathbf{A} \to \mathbf{A}.$$

- A model of **Met**-enriched Lawvere theory [Pow99] is a quantitative algebra with possibly partial, infinitary, nonexpansive operations (=[FMS21]):

$$F(\mathsf{op} : \mathbf{2}_{0.5} \to 1) : \mathbf{A}^{\mathbf{2}_{0.5}} = \mathbf{Met}(\mathbf{2}_{0.5}, \mathbf{A}) \to \mathbf{A}.$$

  Any quantitative equation can be expressed in the theory.
- A model of a discrete [Pow05; HP06] **Met**-enriched Lawvere theory is a quantitative algebra in the sense of [MPP16]. Only discrete quantitative equations ($\mathbf{X}_\top \vdash s =_\varepsilon t$) can be expressed.
- A model of a discrete [Ros24] **Met**-enriched Lawvere theory is a quantitative algebra in the sense of [MPP16], and all and only quantitative equations can be expressed.
- A model of a **Poset**-Lawvere theory for **Set** [NP09] is quantitative algebra with partial, finitary, not necessarily nonexpansive operations (=[Adá+21]).

Recovering Birkhoff's Equational Logic

- With $\mathsf{L} = \{\top\}$, $\mathsf{L}\mathbf{Spa} = \mathbf{Set}$, and all the quantitative equations $\mathbf{X} \vdash s =_\varepsilon t$ are provable by TOP. The remaining fragment of QEL is Birkhoff's logic (3.70).

Recovering Birkhoff's Equational Logic

- With $\mathsf{L} = \{\top\}$, $\mathsf{L}\mathbf{Spa} = \mathbf{Set}$, and all the quantitative equations $\mathbf{X} \vdash s =_\varepsilon t$ are provable by TOP. The remaining fragment of QEL is Birkhoff's logic (3.70).
- Over any L, we can translate a classical equation $X \vdash s = t$ into a quantitative equation $\mathbf{X}_\top \vdash s = t$.
  - This translation *preserves* provability (3.71).

Lifted Signatures

Definition

Given a signature Σ, a **lifted signature** is an endofunctor $\widehat{\Sigma} : \mathbf{GMet} \to \mathbf{GMet}$ that preserves isometric embeddings and lifts the **Set** endofunctor $\Sigma = X \mapsto \coprod_{\mathsf{op}\,:\,n \in \Sigma} X^n$:

$$\begin{array}{ccc} \mathbf{GMet} & \xrightarrow{\widehat{\Sigma}} & \mathbf{GMet} \\ \downarrow & & \downarrow \\ \mathbf{Set} & \xrightarrow{\Sigma} & \mathbf{Set} \end{array}$$

For every $\mathsf{op} : n \in \Sigma$, we get $L_{\mathsf{op}}(X, d) = (X^n, L_{\mathsf{op}}(d))$, and a $\widehat{\Sigma}$-algebra has nonexpansive operations

$$[\![\mathsf{op}]\!] : (A^n, L_{\mathsf{op}}(d)) \to (A, d).$$

Examples include the product lifting, the tensor lifting, the discrete lifting, the c-Lipschitz lifting.
Equivalently with quantitative equations:

$$\forall (X, d_{\mathbf{X}}) \in \mathbf{GMet}, \forall x, y \in X^n, \quad (X, d_{\mathbf{X}}) \vdash \mathsf{op}(x_1, \ldots, x_n) =_{L_{\mathsf{op}}(d_{\mathbf{X}})(x,y)} \mathsf{op}(y_1, \ldots, y_n).$$

Definition

Given a signature Σ, an **almost lifted signature** is a **GMet** endofunctor $\widehat{\Sigma}$ that preserves isometric embeddings and lifts the **Set** endofunctor Σ up to a monic natural transformation ℓ:

$$\begin{array}{ccc} \mathbf{GMet} & \xrightarrow{\widehat{\Sigma}} & \mathbf{GMet} \\ \downarrow & \overset{\ell}{\swarrow\!\!\!\!\swarrow} & \downarrow \\ \mathbf{Set} & \xrightarrow{\Sigma} & \mathbf{Set} \end{array}$$

Seeing the components $\ell_{\mathbf{X}} : U\widehat{\Sigma}\mathbf{X} \hookrightarrow \Sigma X$ as inclusions, $(\widehat{\Sigma}, \ell)$-algebras now have partial operations.

Example

If each operation op comes with an arity (n, d_{op}), then we have an almost lifted signature (c.f. [FMS21])

$$\widehat{\Sigma}(\mathbf{X}) = \coprod_{\mathsf{op}:n\in\Sigma} \mathbf{X}^{(n,d_{\mathsf{op}})}.$$

On Monadicity

In the thesis, we do not prove monadicity, only $\mathbf{QAlg}(\Sigma, \hat{E}) \cong \mathbf{EM}(\widehat{\mathcal{T}}_{\Sigma,\hat{E}})$ in 3.80. In [MSV23], we prove it essentially as follows:

Theorem
$U_0 : \mathbf{QAlg}(\Sigma) \to \mathsf{L}\mathbf{Spa}$ is strictly monadic.

Proof. Left-adjoint by construction of free algebras, and strictly creates U_0-absolute coequalizers following MacLane.

Theorem
$U_1 : \mathbf{QAlg}(\Sigma, \hat{E}) \to \mathsf{L}\mathbf{Spa}$ is strictly monadic.

Proof. Idem for left adjoint, strictly creates U_1-split coequalizers because U_0 creates them and $\mathbf{QAlg}(\Sigma, \hat{E})$ is closed under images of U_0-split homomorphisms.

Theorem
$U : \mathbf{QAlg}(\Sigma, \hat{E} \cup \hat{E}_{\mathbf{GMet}}) \to \mathbf{GMet}$ is strictly monadic.

Proof. By $\mathbf{GMet}$ being a full reflective subcategory of $\mathsf{L}\mathbf{Spa}$ and $U_1 : \mathbf{QAlg}(\Sigma, \hat{E} \cup \hat{E}_{\mathbf{GMet}}) \to \mathsf{L}\mathbf{Spa}$ is strictly monadic.