Projective Algebras and Quotient Monads

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Abstract

Quotient types are quite common in mathematics but they are rather difficult to implement in a programming language. For instance, one can easily define the type of pairs of integers $\mathsf{Int} \times \mathsf{Int}$, but in order to define the type of rational numbers, one needs to quotient $\mathsf{Int} \times \mathsf{Int}$ by the relation $(p, q) \sim (r, s) \Leftrightarrow ps = rq$, which is easier said than done. In the framework of monadic programming, datatypes are free algebras for a monad. Not all algebras for a monad are free, but they are all quotients of free algebras. An algebra for a monad is said to be *projective* if it is both a quotient and a subalgebra of a free algebra. We show that a natural family of projective algebras can be seen as algebras for a quotient monad. In other words, when a quotienting operation is nice enough that 1) the resulting algebra is a subalgebra of the free algebra and 2) it satisfies some naturality condition, then we obtain a monad that models the quotient type.

1 Introduction

In type theory, a major goal is to reproduce the reasoning behind common mathematical constructions within a type system. For instance, most type systems will deal with product types and sum types that correspond to the well known Cartesian product and disjoint union of sets. In the case of products, one realizes that the projections $\pi_A : A \times B \to A$ and $\pi_B : A \times B \to B$ satisfy a simple universal property that characterizes the product (up to isomorphism), and this property can be encoded in the type system to define product types.

One very powerful idea mathematicians use is quotients. They are very useful to group objects together when they satisfy similar properties to abstract away from these properties and see each group as an object on its own. For instance, the set of positive rationals is obtained by quotienting the set $\mathbb{N} \times \mathbb{N}$ by the relation $(p, q) \sim (r, s) \Leftrightarrow ps = rq$. In simple terms, $(p, q) \sim (r, s)$ exactly when the two pairs represent the same ratio. Thus, in order to manipulate ratios of natural numbers, we do not bother with the several possible representations in $\mathbb{N} \times \mathbb{N}$, and we work with the quotient $\mathbb{Q}^+$.

Even though quotients also satisfy a universal property (e.g. they are coequalizers), the latter is hard to implement in a type system, and modelling quotient types has been a challenge.

One option is to avoid quotients completely and work with setoids instead of sets. A setoid is a set S along with an equivalence relation $\sim$ kept explicit (often with a proof that $\sim$ is an equivalence relation). This is not always satisfactory because equality in S is different from $\sim$, so one alwyas needs to make sure they are doing things that are compatible with $\sim$ (e.g.: a function f must satisfy $x \sim y \implies f(x) = f(y)$). At the other extreme, there is Homotopy Type Theory where quotient types can be constructed as a consequence of the axiom of univalence.

A simple idea that could help with quotienting is canonical representatives. If there is an element of S for each equivalence class of $S/\sim$, say chosen by $s : S/\sim \to S$, then one could think of the image of s as the quotient. For example, the equivalence class of ratios in $\mathbb{Q}^+$ each contain a unique reduced fraction $\frac{p}{q}$ (no prime factors in common between p and q). Thus, we could define a positive rational number as being a reduced fraction of two natural numbers. Unfortunately, this breaks when we want to do arithmetic, namely, if $\frac{p}{q}$ and $\frac{r}{s}$ are reduced, then $\frac{p}{q} \cdot \frac{r}{s} = \frac{pq}{rs}$ may not be reduced.

In the context of monadic programming, types are implemented as free algebras of a monad. This makes it impossible to work directly with quotients because, by definition, they are not free. However, there are some cases where quotients are nice enough that one can pick canonical representatives that are compatible with the monad structure. Such an algebra is called projective. Informally, our main results states that when you can define projective algebras globally, you can actually define a new datatype that models these algebras.

2 Background

In this section, we give the crucial definitions and results about monads and algebras that will be needed in the paper. We also give a couple of conrete examples of projective algebras. We assume the reader is familiar with basic category theory.

2.1 Monads and Algebras

Definition 1 (Monad)**.** A **monad** on a category $\mathbf{C}$ is a triple comprised of an endofunctor $M : \mathbf{C} \to \mathbf{C}$ and two natural transformations $\eta : \mathrm{id}_{\mathbf{C}} \Rightarrow M$ and $\mu : M^2 \Rightarrow M$ called the **unit** and **multiplication** respectively that make (1) and (2) commute.

$$\begin{tikzcd} M \arrow[r, "M\eta"] \arrow[dr, "\mathrm{id}_M"'] & M^2 \arrow[d, "\mu"] & M \arrow[l, "\eta M"'] \arrow[dl, "\mathrm{id}_M"] \\ & M & \end{tikzcd} \tag{1}$$

$$\begin{tikzcd} M^3 \arrow[r, "\mu M"] \arrow[d, "M\mu"'] & M^2 \arrow[d, "\mu"] \\ M^2 \arrow[r, "\mu"'] & M \end{tikzcd} \tag{2}$$

Definition 2 (M–algebra)**.** Let (M, η, μ) be a monad on $\mathbf{C}$, an M**–algebra** is a pair (X, x) consisting of an object X and morphism $x : MX \to X$ in $\mathbf{C}$ such that (3) and (4) commute.

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "\mathrm{id}_X"'] & MX \arrow[d, "x"] \\ & X \end{tikzcd} \tag{3}$$

$$\begin{tikzcd} M^2X \arrow[r, "\mu_X"] \arrow[d, "Mx"'] & MX \arrow[d, "x"] \\ MX \arrow[r, "x"'] & X \end{tikzcd} \tag{4}$$

Definition 3 (Homomorphism)**.** Given two M–algebras (X, x) and (Y, y), an M–algebra **homomorphism** $h : (X, x) \to (Y, y)$ is a morphism $h : X \to Y$ in $\mathbf{C}$ making (5) commute.

$$\begin{tikzcd} MX \arrow[r, "Mh"] \arrow[d, "x"'] & MY \arrow[d, "y"] \\ X \arrow[r, "h"'] & Y \end{tikzcd} \tag{5}$$

For a monad M, the category of M-algebras and their homomorphisms is called the **Eilenberg–Moore** category of M and denoted $\mathrm{EM}(M)$. We denote $U^M : \mathrm{EM}(M) \to \mathbf{C}$ the forgetful functor sending an M-algebra (X, x) to X and a homomorphism to its underlying morphism. This functor has a right adjoint $F^M : \mathbf{C} \to \mathrm{EM}(M)$, called the **free** M**–algebra** functor, it sends an object X to (MX, μ_X), which is an M–algebra by the R.H.S. of (1) and (2).

The image of the free algebra functor F^M can be identified with the **Kleisli** category of M, denoted $\mathrm{Kl}(M)$. The objects of $\mathrm{Kl}(M)$ are objects of $\mathbf{C}$, but morphisms in $\mathrm{Hom}_{\mathrm{Kl}(M)}(A, B)$ are exactly the morphisms in $\mathrm{Hom}_{\mathbf{C}}(A, MB)$. We denote Kleisli morphisms in $\mathrm{Hom}_{\mathrm{Kl}(M)}(A, B)$ by $A \rightsquigarrow B$. Kleisli composition is denoted $\circ_M$ and defined by $f \circ_M g = \mu_C \circ Mg \circ f$ for $A \overset{f}{\rightsquigarrow} B \overset{g}{\rightsquigarrow} C$, and the identity Kleisli morphism is $\eta_A : A \rightsquigarrow A$.

The universal property of free algebras states that for any morphism $f : A \to MB$, there is a unique M–algebra homomorphism $h : (MA, \mu_A) \to (MB, \mu_B)$ such that $h \circ \eta_A = f$. The naturality of η and the R.H.S. of (1) implies that $h = \mu_B \circ Mf$. We infer that sending A to (MA, μ_A) and $f : A \rightsquigarrow B$ to the unique $\mu_B \circ Mf$ yields an embedding $\mathrm{Kl}(M) \to \mathrm{EM}(M)$ whose image is precisely the free algebras.

Note that any M–algebra (X, x) is a quotient of the free M–algebra on X, indeed (4) can be seen as stating $x : (MX, \mu_X) \to (X, x)$ is a homomorphism, and x is epic because it has a right inverse η_X. If (X, x) is also a subalgebra of $F^M X$, we say it is projective.

Definition 4 (Projective algebra)**.** An M–algebra (X, x) is called a **projective algebra** if there is a monic homomorphism $h : (X, x) \hookrightarrow (MX, \mu_X)$. We obtain the following commutative diagram.

$$\begin{tikzcd}
MX \arrow[r, "Mh"] \arrow[d, "x"'] & MMX \arrow[r, "Mx"] \arrow[d, "\mu_X"'] & MX \arrow[d, "x"] \\
X \arrow[r, hook, "h"'] & MX \arrow[r, two heads, "x"'] & X
\end{tikzcd} \tag{6}$$

2.2 (Non)-Examples of Projective Algebras

Let $\mathbf{Ab}$ be the category of abelian groups, it has a straightforward forgetful functor U to the category of sets (forget the group opertaion). This functor has a right adjoint F and the composite FU is a monad, that we write $T_{\mathbf{Ab}}$, such that $\mathbf{Ab} \cong \mathrm{EM}(T_{\mathbf{Ab}})$.

Let $P = \{2, 3, 5, 7, 11, \dots\}$ be the set of prime numbers, one can see $\mathbb{Q}^+$, the set of positive rational numbers, as the free abelian group on P. Namely, the multiplicative group $\mathbb{Q}^+$ is the image of P under the free algebra functor $F^{T_{\mathbf{Ab}}} : \mathbf{Set} \to \mathrm{EM}(T_{\mathbf{Ab}})$. We will exhibit one quotient of $\mathbb{Q}^+$ that is projective and one that is not.

Let $K = \langle p \in P \setminus \{2\} \rangle$ be the subgroup of $\mathbb{Q}^+$ generated by all the primes except 2. Quotienting by K yields the additive group of integers $\mathbb{Z}$. Indeed, each equivalence class of $\mathbb{Q}^+/K$ has a single element of the form 2^z with $z \in \mathbb{Z}$ and multiplying $[2^z]$ with $[2^{z'}]$ yields $[2^{z+z'}]$ by a standard property of exponents. Furthermore, sending $z \in Z$ to $2^z \in \mathbb{Q}^+$ is a group homomorphism (by the same property of exponents), thus $\mathbb{Z}$ is a projective $T_{\mathbf{Ab}}$–algebra as witnessed by the aforementionned homomorphisms

$$\mathbb{Q}^+ \to \mathbb{Q}^+/K \cong \mathbb{Z} \to \mathbb{Q}^+.$$

If we further quotient by the subgroup $2\mathbb{Z}$ to obtain the two element group, we note that there can be no homomorphism $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Q}^+$ because no positive rational has order 2 like $1 \in \mathbb{Z}/2\mathbb{Z}$. In fact, since any free abelian group has no element of order 2, we conclude that $\mathbb{Z}/2\mathbb{Z}$ cannot be a projective $T_{\mathbf{Ab}}$–algebra.

Our first example can be made quite more general by noting that $\mathbb{Z}$ is the free abelian group on a singleton. Let M be a monad on **Set** that preserves injections and surjections, if $f : X \to Y$ is a surjective function and $g : Y \to X$ is its right inverse, one can check that (7) is commutative, hence (MX, μ_X) is a projective algebra.

$$
\begin{tikzcd}
MMX \arrow[r, "MMf"] \arrow[d, "\mu_X"'] & MMY \arrow[r, "MMg"] \arrow[d, "\mu_Y"'] & MMX \arrow[d, "\mu_X"] \\
MX \arrow[r, "Mf"'] & MY \arrow[r, "Mg"'] & MX
\end{tikzcd}
\tag{7}
$$

This is not very interesting as any free algebra is already a quotient and subalgebra of itself (take $f = \mathrm{id}$), hence is projective. Nonetheless, we can extract a bit of intuition by asking what general condition we should put on Mf and Mg to obtain a projective algebra. The option we will explore is split idempotents.

Definition 5 (Idempotents)**.** Let $\mathbf{C}$ be a category, a "morphism" $f : A \to A$ in $\mathbf{C}$ is called **idempotent** if $f \circ f = f$. It is called **split idempotent** if there exist morphisms $s : E \to A$ and $r : A \to E$ such that $s \circ r = f$ and $r \circ s = \mathrm{id}_E$. We can show that split idempotents are idempotent by

$$f \circ f = s \circ r \circ s \circ r = s \circ \mathrm{id}_E \circ r = f.$$

We call E the **splitting** of the idempotent. It is clear (because of their inverses) that r is epic and s is monic.

This yields a very simple way to find projective M–algebras. If $f : (MX, \mu_X) \to (MX, \mu_X)$ is a split idempotent in $\mathrm{EM}(M)$, the splitting will be a projective algebra as shown below.

$$
\begin{tikzcd}
MMX \arrow[r, "Mr"] \arrow[d, "\mu_X"'] \arrow[rr, bend left, "Mf"] & ME \arrow[r, "Ms"] \arrow[d, "e"'] & MMX \arrow[d, "\mu_X"] \\
MX \arrow[r, two heads, "r"'] \arrow[rr, bend right, "f"'] & E \arrow[r, hook, "s"'] & MX
\end{tikzcd}
\tag{8}
$$

Let us give a concrete example of this with the monad for pointed semilattices.

Definition 6 (Pointed semilattice)**.** A **pointed semilattice** is a set X equipped with a binary operation $\oplus : X \times X \to X$ (written infix) and a distinguised element (called **point**) $\star \in X$ such that for any $x, y, z \in X$:

$$
\begin{aligned}
x \oplus x &= x && \text{idempotence} \\
x \oplus y &= y \oplus x && \text{commutativity} \\
x \oplus (y \oplus z) &= (x \oplus y) \oplus z && \text{associativity}
\end{aligned}
$$

A homomorphism of pointed semilattices $(X, \oplus_X, \star_X) \to (Y, \oplus_Y, \star_Y)$ is a function $f : X \to Y$ that commutes with the operation and the point, i.e.: $\forall x, x' \in X, f(x \oplus_X x') = f(x) \oplus_Y f(x')$ and $f(\star_X) = \star_Y$. We denote **PSLat** the category of pointed semilattices.

One can show that the free pointed semilattice on X is the finite powerset of $X + \mathbf{1} := X \sqcup \{\star\}$ with the operation being union and the point being the singleton $\{\star\}$.

Let $K : \mathcal{P}(X + \mathbf{1}) \to \mathcal{P}(X + \mathbf{1})$ be defined by sending a finite $S \subseteq X + \mathbf{1}$ to $S \cup \{\star\}$. This is idempotent because adding $\star$ twice is the same thing as adding it once. Moreover, K is a homomorphism (essentially) because it commutes with union of sets and sends $\{\star\}$ to itself.

Following our thread of split idempotents, one checks that in **Set**, any idempotent $f : A \to A$ is split by considering the image of f: $A \twoheadrightarrow \mathrm{Im}(f) \hookrightarrow A$. Hence, it remains to check the splitting of K is a pointed semilattice.

The image of K contains all the sets that contain $\star$, and we note the union of two such sets still contains $\star$ and the point $\{\star\}$ is in the image of K. Therefore, $\mathrm{Im}(K)$ is indeed a pointed semilattice and moreover it is a quotient and subalgebra of $\mathcal{P}(X + \mathbf{1})$, so it is projective. This is our first example of a projective and not free algebra.

Actually, this argument works for any set X, so we have an idempotent K_X and a projective algebra $\mathrm{Im}(K_X)$ for every X, and this family is natural (in the categorical sense that we develop below). Our main result states that this is enough to obtain a monad structure on the assignment $X \mapsto \mathrm{Im}(K_X)$. In this case, one can identify the resulting monad with $\mathcal{P} + \mathbf{1} = X \mapsto \mathcal{P}X + \mathbf{1}$, which is the free semilattice with bottom monad (a semilattice with bottom is a pointed semilattice satisfying the bottom equation: $x \oplus \star = x$).

3 Main Result

Let us generalize the idea above. For any object X in $\mathbf{C}$, we want an idempotent endomorphism on the free algebra (MX, μ_X). Since the Kleisli category $KL(M)$ contains precisely the free algebras, equivalently, we want a Kleisli morphism $K_X : X \rightsquigarrow X$ that is idempotent. Moreover, we want this family to be natural in X, thus we need an idempotent natural transformation $K : \mathrm{id}_{\mathrm{Kl}(M)} \Rightarrow \mathrm{id}_{\mathrm{Kl}(M)}$.

Remark 7. We will soon see that this K induces an idempotent natural transformation $\overline{K} : M \Rightarrow M$ and this may be a better starting point because it is enough for our purposes. However, since $\overline{K}$ does not induce the K described above, we still start from K and at some point we only use $\overline{K}$.

Let us show some properties of K. First, idempotence says that $K_X \circ_M K_X = K_X$, or equivalently,

$$\mu_X \circ M(K_X) \circ K_X = K_X. \tag{9}$$

Next, we can apply naturality of K to different morphisms in $\mathrm{Kl}(M)$ to obtain different identities.

$$\begin{tikzcd} MX \arrow[r, squiggly, "\mathrm{id}_{MX}"] \arrow[d, squiggly, "K_{MX}"'] & X \arrow[d, squiggly, "K_X"] \\ MX \arrow[r, squiggly, "\mathrm{id}_{MX}"'] & X \end{tikzcd} \quad (10) \qquad \begin{tikzcd} MMX \arrow[r, squiggly, "\mu_X"] \arrow[d, squiggly, "K_{MMX}"'] & X \arrow[d, squiggly, "K_X"] \\ MMX \arrow[r, squiggly, "\mu_X"'] & X \end{tikzcd} \quad (11) \qquad \begin{tikzcd} MX \arrow[r, squiggly, "Mf"] \arrow[d, squiggly, "K_{MX}"'] & Y \arrow[d, squiggly, "K_Y"] \\ MX \arrow[r, squiggly, "Mf"'] & Y \end{tikzcd} \quad (12)$$

Now, recall that these diagrams live in $\mathrm{Kl}(M)$, thus, the following equations are derived from each diagram.

$$\mu_X \circ M(K_X) \circ \mathrm{id}_{MX} = \mu_X \circ M(\mathrm{id}_{MX}) \circ K_{MX} \tag{10}$$
$$\mu_X \circ M(K_X) \circ \mu_X = \mu_X \circ M(\mu_X) \circ K_{MMX} \tag{11}$$

$$\mu_Y \circ M(K_Y) \circ Mf = \mu_Y \circ MMf \circ K_{MX} \tag{12}$$

From this, we will construct a monad M^K whose free algebra on X is the image of K applied to the free M–algebra on X. We will also construct a monad map $M \Rightarrow M^K$ expressing M^K as a quotient of M.

Proposition 8. *Defining* $\overline{K}_X : MX \to MX = \mu_X \circ M(K_X)$*, we can show that* $\overline{K}_X$ *is idempotent.*

Proof. We show that $\mu_X \circ M(K_X) \circ \mu_X \circ M(K_X) = \mu_X \circ M(K_X)$ by paving the following diagram.

$$\begin{tikzcd}
MX \arrow[rr, "M(K_X)"] \arrow[d, "M(K_X)"'] \arrow[dr, phantom, "(a)"] & & MMX \arrow[dd, "\mu_X"] \\
MMX \arrow[r, "MM(K_X)"] \arrow[d, "\mu_X"'] \arrow[dr, phantom, "(b)"] & MMMX \arrow[ur, "M(\mu_X)"] \arrow[d, "\mu_{MX}"] \arrow[r, phantom, "(c)"] & {} \\
MX \arrow[r, "M(K_X)"'] & MMX \arrow[r, "\mu_X"'] & MX
\end{tikzcd} \tag{13}$$

(a) Apply M to (9).

(b) Naturality of μ.

(c) Associativity of μ.

∎

Furthermore, since this $\overline{K}_X$ is the image of K_X under the embedding $\mathrm{Kl}(M) \to \mathrm{EM}(M)$, we obtain that $\overline{K}_X$ is an M–algebra endomorphism on (MX, μ_X). This is restated and proven for completeness below.

Lemma 9. *For any X, we have* $\overline{K}_X \circ \mu_X = \mu_X \circ M(\overline{K}_X)$.

Proof. We pave the following diagram.

$$\begin{tikzcd}
MMX \arrow[rr, "M(\overline{K}_X)"] \arrow[dr, "MM(K_X)"'] \arrow[ddd, "\mu_X"'] & {} \arrow[d, phantom, "(a)"] & MMX \arrow[ddd, "\mu_X"] \\
{} \arrow[r, phantom, "(b)"] & MMMX \arrow[ur, "M(\mu_X)"'] \arrow[d, "\mu_{MX}"] \arrow[r, phantom, "(c)"] & {} \\
& MMX \arrow[dr, "\mu_X"] \arrow[d, phantom, "(d)"] & \\
MX \arrow[ur, "M(K_X)"] \arrow[rr, "\overline{K}_X"'] & {} & MX
\end{tikzcd} \tag{14}$$

(a) Def of $\overline{K}_X$ and functoriality of M.

(b) Naturality of μ.

(c) Associativity of μ.

(d) Def of $\overline{K}_X$.

∎

Next, we would like $\overline{K}_X$ to be split and define $M^K(X)$ as the splitting of $\overline{K}_X$. One way to formulate this property which will be easier to work with is stated in the following lemma.

Lemma 10. *An idempotent morphism $f : A \to A$ in $\mathbf{C}$ is split if and only if the equalizer of f and id_A exists.*

Now, if every idempotent $\overline{K}_X$ is split, we can define a subfunctor M^K of M as follows. For $X \in \mathbf{C}_0$, let $M^K X$ be the equalizer of $\overline{K}_X, \mathrm{id}_{MX} : MX \to MX$. Namely, there is (a monic) $\iota_X : M^K X \to MX$ satisfying $\overline{K}_X \circ \iota_X = \iota_X$ such that for any $e : Y \to MX$ satisfying $\overline{K}_X \circ e = e$, there is a unique morphism $! : Y \to M^K X$ making (15) commute.

$$\begin{tikzcd} Y \arrow[d, dashed, "!"'] \arrow[dr, "e"] & & \\ M^K X \arrow[r, "\iota_X"'] & MX \arrow[r, shift left, "\overline{K}_X"] \arrow[r, shift right, "\mathrm{id}_{MX}"'] & MX \end{tikzcd} \tag{15}$$

In order to give the action of M^K on morphisms, we need the following lemma.

Lemma 11. *$\overline{K}$ is a natural transformation $M \Rightarrow M$.*

Proof. We need to show that for any $f : X \to Y$, $\overline{K}_Y \circ Mf = Mf \circ \overline{K}_X$. We have the following derivation.

$$\begin{aligned} \overline{K}_Y \circ Mf &= \mu_Y \circ M(K_Y) \circ Mf && \text{def. } \overline{K} \\ &= \mu_Y \circ MMf \circ K_{MX} && \text{by (12)} \\ &= Mf \circ \mu_X \circ K_{MX} && \text{naturality of } \mu \\ &= Mf \circ \mu_X \circ M(K_X) && \text{by (10)} \\ &= Mf \circ \overline{K}_X && \text{def. } \overline{K} \end{aligned}$$

□

Remark 12. From this point, we do not have to use any hypothesis about K. Thus, starting with an idempotent natural transformation $\overline{K} : M \Rightarrow M$ such that $\overline{K}_X : MX \to MX$ is split and an M–algebra homomorphism (with the free algebra structure on MX), we can develop the rest of the section. Another very close starting point will be used in the application section.

Now, for any $f : X \to Y$, we know that both squares on the R.H.S. of diagram (16) commute (id is trivially a natural transformation).

$$\begin{tikzcd} M^K X \arrow[r, "\iota_X"] \arrow[d, dashed, "M^K f"'] & MX \arrow[r, shift left, "\overline{K}_X"] \arrow[r, shift right, "\mathrm{id}_{MX}"'] \arrow[d, "Mf"'] & MX \arrow[d, "Mf"] \\ M^K Y \arrow[r, "\iota_Y"'] & MY \arrow[r, shift left, "\overline{K}_Y"] \arrow[r, shift right, "\mathrm{id}_{MY}"'] & MY \end{tikzcd} \tag{16}$$

From this, we can infer that $Mf \circ \iota_X$ equalizes $\overline{K}_Y$ and id_{MY}. Indeed, we have

$$\begin{aligned} \overline{K}_Y \circ Mf \circ \iota_X &= Mf \circ \overline{K}_X \circ \iota_X \\ &= Mf \circ \mathrm{id}_{MX} \circ \iota_X \\ &= \mathrm{id}_{MY} \circ Mf \circ \iota_X. \end{aligned}$$

Then, from the universality of $M^K Y$, there is a unique morphism $M^K f : M^K X \to M^K Y$ making (16) commute. The uniqueness of $M^K f$ in (16) also shows that $M^K(f \circ g) = M^K f \circ M^K g$, thus M^K is a functor $\mathbf{C} \to \mathbf{C}$.

Proposition 13. *The family $\{\iota_X \mid X \in \mathbf{C}_0\}$ is a natural transformation $\iota : M^K \Rightarrow M$ with monic components, so M^K is a subfunctor of M.*

Proof. The naturality follows trivially from the commutativity of left square in (16). The monicity comes from the standard result that equalizers are monic. □

Observe that by idempotence, $\overline{K}_X$ also equalizes $\overline{K}_X$ and id_{MX}, so we get the following diagram.

$$\begin{tikzcd} MX \arrow[d, dashed, "\widehat{K}_X"'] \arrow[dr, "\overline{K}_X"] & & \\ M^K X \arrow[r, "\iota_X"'] & MX \arrow[r, shift left, "\overline{K}_X"] \arrow[r, shift right, "\mathrm{id}_{MX}"'] & MX \end{tikzcd} \tag{17}$$

In the sequel, we will denote by $\widehat{K}_X$ the unique morphism satisfying $\iota_X \circ \widehat{K}_X = \overline{K}_X$.

Lemma 14. *For any $X \in \mathbf{C}_0$, $\widehat{K}_X \circ \iota_X = \mathrm{id}_{M^K X}$.*

Proof. Using the definitions of $\widehat{K}_X$ and ι_X, we have

$$\iota_X \circ \widehat{K}_X \circ \iota_X = \overline{K}_X \circ \iota_X = \mathrm{id}_{MX} \circ \iota_X = \iota_X \circ \mathrm{id}_{M^K X}.$$

The lemma follows by monicity of ι_X. □

Remark 15. One way to summarize this is to say that $M^K X$ is the splitting of $\overline{K}_X$ with $\iota_X \circ \widehat{K}_X$ being the monic-epic factorization of $\overline{K}_X$.

Proposition 16. *The family $\{\widehat{K}_X \mid X \in \mathbf{C}_0\}$ is a natural transformation with epic components $\widehat{K} : M \Rightarrow M^K$.*

Proof. First, we claim that for any $f : X \to Y$, $M^K f \circ \widehat{K}_X = \widehat{K}_Y \circ Mf$. We have the following derivation.

$$\begin{aligned} \iota_Y \circ M^K f \circ \widehat{K}_X &= Mf \circ \iota_X \circ \widehat{K}_X && \text{naturality of } \iota \\ &= Mf \circ \overline{K}_X && \text{def of } \widehat{K}_X \\ &= \overline{K}_Y \circ Mf && \text{naturality of } \overline{K} \\ &= \iota_Y \circ \widehat{K}_Y \circ Mf && \text{def of } \widehat{K}_Y \end{aligned}$$

The claim follows since ι_Y is a monomorphism. The components $\widehat{K}_X$ are epimorphisms because they have ι_X as a right inverse by Lemma 14. □

3.1 Monadicity of M^K

Next, we want to show that M^K is a monad with unit $\eta^K := \widehat{K} \cdot \eta$ and multiplication $\mu^K := \widehat{K} \cdot \mu \cdot (\iota \diamond \iota)$.[1] We divide the proof in multiple lemmas.

Lemma 17. *For any $X \in \mathbf{C}_0$, $\mu_X \circ M(\iota_X) = \overline{K}_X \circ \mu_X \circ M(\iota_X)$.*

[1] We write $\diamond$ for the horizontal composition of natural tranformations.

Proof. We pave the following diagram.

$$
\begin{tikzcd}
& MMX \arrow[rr, "\mu_X"] \arrow[dd, "M(\overline{K}_X)"'] \arrow[dr, "MM(K_X)"] & {} \arrow[d, phantom, "(c)"] & MX \arrow[d, "M(K_X)"] \\
MM^KX \arrow[ur, "M(\iota_X)"] \arrow[dr, "M(\iota_X)"'] \arrow[r, phantom, "(a)"] & {} \arrow[r, phantom, "(b)"] & MMMX \arrow[r, "\mu_{MX}"'] \arrow[dl, "M(\mu_X)"] & MMX \arrow[d, "\mu_X"] \\
& MMX \arrow[rr, "\mu_X"'] & {} \arrow[u, phantom, "(d)"] & MX
\end{tikzcd}
\tag{18}
$$

(a) By defintion of M^K.

(b) Functoriality of M and def of $\overline{K}_X$.

(c) By naturality of μ.

(d) Associativity of μ.

□

Lemma 18. *Anagolously to* (10), *we also have* $\mu_X \circ M(\overline{K}_X) = \mu_X \circ \overline{K}_{MX}$.

Proof. We pave the following diagram.

$$
\begin{tikzcd}
MMX \arrow[r, "MM(K_X)"] \arrow[rr, bend left, "M(\overline{K}_X)"] \arrow[dr, "M(K_{MX})"'] \arrow[dd, "\overline{K}_{MX}"'] & MMMX \arrow[r, "M(\mu_X)"] \arrow[d, phantom, "(a)"] & MMX \arrow[dd, "\mu_X"] \\
{} \arrow[r, phantom, "(b)"] & MMMX \arrow[ur, "M(\mu_X)"'] \arrow[dl, "\mu_{MX}"] \arrow[dr, phantom, "(c)"] & {} \\
MMX \arrow[rr, "\mu_X"'] & & MX
\end{tikzcd}
\tag{19}
$$

(a) Apply M to (10).

(b) Def of $\overline{K}$.

(c) Associativity of μ.

□

Now we can prove one side of the unit diagram for the monad M^K commutes.

Lemma 19. *For any* $X \in \mathbf{C}_0$, $\mu^K_X \circ M^K(\eta^K_X) = \mathrm{id}_{M^K X}$.

Proof. We will show that $\iota_X \circ \mu^K_X \circ M^K(\eta^K_X) = \iota_X$ from which the result follows by monicity of ι_X. We pave the following diagram.

$$
\begin{tikzcd}
M^KX \arrow[r, "M^K(\eta_X)"] \arrow[d, dashed, "\iota_X"'] \arrow[rr, bend left, "M^K(\eta^K_X)"] \arrow[dddd, bend right=40, "\iota_X"'] \arrow[dr, phantom, "\text{(a)}"] & M^KMX \arrow[r, "M^K(\widehat{K}_X)"] \arrow[d, dashed, "\iota_{MX}"] \arrow[dr, phantom, "\text{(b)}"] & M^KM^K \arrow[d, dashed, "\iota_{M^KX}"] \arrow[dddd, bend left=40, "\mu^K_X"] \\
MX \arrow[r, "M(\eta_X)"] \arrow[dd, "\mathrm{id}_{MX}"] \arrow[d, phantom, "\text{(c)}"] & MMX \arrow[r, "M(\widehat{K}_X)"] \arrow[d, "M(\overline{K}_X)"] \arrow[ddl, "\mu_X"] \arrow[r, phantom, "\text{(d)}"'] & MM^KX \arrow[dl, "M(\iota_X)"'] \arrow[d, "M(\iota_X)"] \arrow[d, phantom, "\text{(e)}"'] \\
 & MMX \arrow[ddl, "\mu_X"] \arrow[dl, phantom, "\text{(g)}"] \arrow[dr, phantom, "\text{(h)}"] & MMX \arrow[l, "M(\overline{K}_X)"'] \arrow[d, dashed, "\mu_X"] \\
MX \arrow[d, "\overline{K}_X"] \arrow[u, phantom, "\text{(f)}"] & & MX \arrow[d, "\widehat{K}_X"] \arrow[dll, "\overline{K}_X"'] \arrow[dl, phantom, "\text{(i)}"] \\
MX & & M^KX \arrow[ll, "\iota_X"]
\end{tikzcd}
\tag{20}
$$

(a) Naturality of ι.

(b) Naturality of ι.

(c) Monadicity of (M, η, μ).

(d) Apply M to $\overline{K}_X = \iota_X \circ \widehat{K}_X$.

(e) Apply M to $\overline{K}_X \circ \iota_X = \iota_X$.

(f) $\overline{K}_X \circ \iota_X = \iota_X$.

(g) Lemma 9.

(h) Lemma 9.

(i) Def of $\widehat{K}$.

□

Now for the other side of the unit diagram.

Lemma 20. *For any $X \in \mathbf{C}_0$, $\mu^K_X \circ \eta^K_{M^KX} = \mathrm{id}_{M^KX}$.*

Proof. Alternatively, we pave the following diagram.

$$
\begin{tikzcd}
M^KX \arrow[rr, "\eta_{M^KX}"] \arrow[dddd, "\mathrm{id}_{M^KX}"'] \arrow[ddr, "\iota_X"] \arrow[ddrr, phantom, "\text{(a)}"] & & MM^KX \arrow[rr, "\widehat{K}_{M^KX}"] \arrow[dd, "M(\iota_X)"'] \arrow[ddr, "M(\iota_X)"] \arrow[ddrr, phantom, "\text{(b)}"] & & M^KM^KX \arrow[dd, "M^K(\iota_X)"'] \arrow[dddd, bend left, equal, "\iota \diamond \iota"] \\
 & & & & \\
 & MX \arrow[r, "\eta_{MX}"] \arrow[ddr, "\mathrm{id}_{MX}"] \arrow[ddl, phantom, "\text{(c)}"] \arrow[ddr, phantom, "\text{(d)}"'] & MMX \arrow[dd, "\mu_X"] \arrow[ddr, phantom, "\text{(e)}"] & MMX \arrow[r, "\widehat{K}_{MX}"] \arrow[d, "M(\overline{K}_X)"'] \arrow[ddr, "\overline{K}_{MX}"] \arrow[r, phantom, "\text{(f)}"'] & M^KMX \arrow[dd, "\iota_{MX}"] \\
 & & & MMX \arrow[dl, "\mu_X"] \arrow[d, phantom, "\text{(g)}"] & \\
M^KX & & MX \arrow[ll, "\widehat{K}_X"] & & MMX \arrow[ll, "\mu_X"]
\end{tikzcd}
\tag{21}
$$

(a) Naturality of η.

(b) Naturality of $\widehat{K}$.

(c) Lemma 14.

(d) Monadicity of (M, μ, η).

(e) ι_X equalizes $\overline{K}_X$ and id_{MX}.

(f) Definition of $\widehat{K}_{MX}$.

(g) Lemma 18.

$\square$

Lastly, we show that μ^K is associative.

Lemma 21. *For any $X \in \mathbf{C}_0$, $\mu^K \circ M^K\mu^K = \mu^K \circ \mu^K M^K$.*

Proof. We pave the following diagram.

$$\begin{tikzcd}
M^KM^KM^K \arrow[r, "M^K\iota M^K"] \arrow[d, "\iota M^KM^K"'] \arrow[rr, bend left, "M^K(\iota\diamond\iota)"] \arrow[dd, bend right=60, "(\iota\diamond\iota)M^K"'] \arrow[dr, phantom, "\text{(a)}"] & M^KMM^K \arrow[r, "M^KM\iota"] \arrow[d, "\iota MM^K"'] \arrow[dr, phantom, "\text{(b)}"] & M^KMM \arrow[r, "M^K\mu"] \arrow[d, "\iota MM"'] \arrow[dr, phantom, "\text{(c)}"] & M^KM \arrow[r, "M^K\widehat{K}"] \arrow[d, "\iota M"'] \arrow[dr, "M^K\overline{K}"'] \arrow[dr, phantom, "\text{(d)}"] & M^KM^K \arrow[d, "M^K\iota"] \arrow[dd, bend left=60, "\iota\diamond\iota"] \\
MM^KM^K \arrow[r, "M\iota M^K"'] \arrow[d, "M\iota M^K"'] & MMM^K \arrow[r, "MM\iota"'] \arrow[dr, "MM\iota"'] \arrow[dddd, "\mu M^K"'] & MMM \arrow[r, "M\mu"'] \arrow[d, phantom, "\text{(f)}"] & MM \arrow[dr, "M\overline{K}"'] \arrow[r, phantom, "\text{(e)}"] & M^KM \arrow[d, "\iota M"] \\
MMM^K \arrow[d, "\mu M^K"'] \arrow[r, phantom, "\text{(g)}"] & & MMM \arrow[rr, "M\mu"'] \arrow[dd, "\mu M"'] \arrow[l, phantom, "\text{(h)}"] \arrow[dr, phantom, "\text{(i)}"] & & MM \arrow[d, "\mu"] \\
MM^K \arrow[d, "\widehat{K}M^K"'] \arrow[dr, "\overline{K}M^K"] \arrow[d, phantom, "\text{(j)}"] & & & & M \arrow[d, "\widehat{K}"] \arrow[dl, equal] \\
M^KM^K \arrow[r, "\iota M^K"'] \arrow[rr, bend right, "\iota\diamond\iota"'] & MM^K \arrow[r, "M\iota"'] & MM \arrow[r, "\mu"'] & M \arrow[r, "\widehat{K}"'] & M^K
\end{tikzcd}$$

(a) Naturality of ι.

(b) Naturality of ι.

(c) Naturality of ι.

(d) Def of $\widehat{K}$.

(e) Naturality of ι.

(f) Lemma 17 acted on the left by M.

(g) Lemma 17 acted on the right by M^K.

(h) Naturality of μ.

(i) Associativity of μ.

(j) Def of $\widehat{K}_{M^KX}$.

$\square$

Theorem 22. *The triple (M^K, η^K, μ^K) is a monad.*

Proof. We have to show the following diagrams commute.

$$\begin{tikzcd}
M^K \arrow[r, Rightarrow, "M\eta^K"] \arrow[dr, Rightarrow, "\mathbb{1}_{M^K}"'] & (M^K)^2 \arrow[d, Rightarrow, "\mu^K"] & M^K \arrow[l, Rightarrow, "\eta^K M^K"'] \arrow[dl, Rightarrow, "\mathbb{1}_{M^K}"] \\
& M^K &
\end{tikzcd} \tag{22}$$

$$\begin{tikzcd}
(M^K)^3 \arrow[r, Rightarrow, "M^K\mu^K"] \arrow[d, Rightarrow, "\mu^K M^K"'] & (M^K)^2 \arrow[d, Rightarrow, "\mu^K"] \\
(M^K)^2 \arrow[r, Rightarrow, "\mu^K"'] & M
\end{tikzcd} \tag{23}$$

Lemmas 19, 20 and 21 respectively show the commutativity of the L.H.S. of (22), the R.H.S. of (22) and (23). $\square$

3.2 Relating M–algebras and M^K–algebras

We already have natural transformations ι and $\widehat{K}$ between M and M^K in both directions, but it is not enough to relate their algebras. For that, we would need for ι and $\widehat{K}$ to be monad maps. Unfortunatley, while $\widehat{K}$ is a monad map as shown below, we have to proceed differently for the other direction.

Theorem 23. *The natural transformation $\widehat{K} : M \Rightarrow M^K$ is a monad map.*

Proof. We have to show the following diagrams commute.

$$\begin{tikzcd}
\mathrm{id}_{\mathbf{C}} \arrow[r, Rightarrow, "\eta"] \arrow[dr, Rightarrow, "\eta^K"'] & M \arrow[d, Rightarrow, "\widehat{K}"] \\
& M^K
\end{tikzcd} \tag{24}$$

$$\begin{tikzcd}
M^2 \arrow[r, Rightarrow, "\widehat{K}\circ\widehat{K}"] \arrow[d, Rightarrow, "\mu"'] & (M^K)^2 \arrow[d, Rightarrow, "\mu^K"] \\
M \arrow[r, Rightarrow, "\widehat{K}"'] & M^K
\end{tikzcd} \tag{25}$$

(24) is trivial because that is the definition of η^K. For (25), we pave the following diagram.

$$\begin{tikzcd}
MMX \arrow[rrr, "\widehat{K}_{MX}"] \arrow[rrrrr, Rightarrow, bend left, "\widehat{K}\circ\widehat{K}"] \arrow[drrr, "\overline{K}_{MX}"] \arrow[ddrr, "\overline{K}_{MX}"] \arrow[dddd, "\mu_X"'] \arrow[drr, phantom, "(a)"] \arrow[ddr, phantom, "(c)"] & & & M^K MX \arrow[rr, "M^K(\widehat{K}_X)"] \arrow[d, "\iota_{MX}"] \arrow[drr, "M^K(\overline{K}_X)"] \arrow[dr, phantom, "(b)"] \arrow[ddr, phantom, "(d)"] & & \mu^K_X \arrow[d, "M^K(\iota_X)"] \arrow[dddd, bend left, "\mu^K_X"] \\
& & & MMX \arrow[dl, "\overline{K}_{MX}"'] \arrow[drr, "M(\overline{K}_X)"] \arrow[ddr, phantom, "(f)"] & & M^K MX \arrow[d, "\iota_{MX}"] \\
& (e) & MMX \arrow[d, "\mu_X"] \arrow[drrr, "\mu_X"] & & & MMX \arrow[d, "\mu_X"] \\
& & MX \arrow[drrr, "\widehat{K}_X"] \arrow[rr, phantom, "(g)"] \arrow[d, phantom, "(h)"] & & & MMX \arrow[d, "\widehat{K}_X"] \\
MX \arrow[uur, "\overline{K}_X"'] \arrow[rrrrr, "\widehat{K}_X"'] & & & & & M^K X
\end{tikzcd} \tag{26}$$

(a) Def of $\widehat{K}_{MX}$.

(b) Def of $\widehat{K}_X$.

(c) Idempotence of $\widehat{K}_{MX}$.

(d) Naturality of ι.

(e) Lemmas 9 and 18.

(f) Lemma 18.

(g) Paths are equal.

(h) Lemma 14 and $\overline{K}_X = \iota_X \circ \widehat{K}_X$.

□

From a standard result, we obtain a functor $U^K : \mathrm{EM}(M^K) \to \mathrm{EM}(M)$ that sends an algebra (A, α) to $(A, \alpha \circ \widehat{K}_A)$ and acts trivially on morphisms. It is fully faithful because $\widehat{K}$ has epic components.

To go in the other direction, our first attempt was to use the embedding $\iota : M^K \Rightarrow M$ in the following way. Given an M–algebra $\alpha : MA \to A$, we expected that the composition $M^K A \xrightarrow{\iota_A} MA \xrightarrow{\alpha} a$ was the natural M^K–algebra on A corresponding to α.

However, in general $\alpha \circ \iota_A$ is not an M^K–algebra because it might not satisfy the unit law, that is,

$$\alpha \circ \iota_A \circ \eta^K = \mathrm{id}_A.$$

In other words, ι is possibly not a monad map (as we will see in the application in to $\mathcal{C}^{\downarrow}$).

3.3 Main Theorem

We summarize our abstract results in this main theorem, the third point will be proven later.

Theorem 24. *Let (M, η, μ) be a monad on $\mathbf{C}$ and assuming one of the following holds:*

- *There is an idempotent natural transformation $K : \mathrm{id}_{\mathrm{Kl}(M)} \Rightarrow \mathrm{id}_{\mathrm{Kl}(M)}$ such that for every $X \in \mathbf{C}_0$, $\overline{K}_X := \mu_X \circ MK_X$ is split.*
- *There is an idempotent natural transformation $\overline{K} : M \Rightarrow M$ such that for every $X \in \mathbf{C}_0$, $\overline{K}_X$ is split and an algebra homomorphism $(MX, \mu_X) \to (MX, \mu_X)$.*
- *There is a natural transformation $K : \mathrm{id}_{\mathbf{C}} \Rightarrow M$ such that for every $X \in \mathbf{C}_0$, $\overline{K}_X := \mu_X \circ MK_X$ is split and idempotent.*

Then, there is a monad (M^K, η^K, μ^K) such that $M^K X$ is the splitting of $\overline{K}_X$ and M^K is a quotient of M.

Remark 25. The three conditions are very close to each other, and in fact the second and third are equivalent, but the first is not. Indeed, the naturality condition in the first condition is stronger than for the other two.

4 (Non-)Examples

Here we list examples and non-examples we encountered while studying this construction.

4.1 Semilattices with Bottom

Defining $\overline{K}_X : \mathcal{P}(X + \mathbf{1}) \to \mathcal{P}(X + \mathbf{1}) = S \mapsto S \cup \{\star\}$, we indeed find that this is an idempotent natural transformation $\mathcal{P}(- + \mathbf{1}) \Rightarrow \mathcal{P}(- + \mathbf{1})$ whose components are homomorphisms (with respect to the free algebra structure). Therefore, we can apply the general construction and one can check that we obtain the monad $\mathcal{P} + \mathbf{1}$ that is presented by semilattices with bottom.

4.2 Convex Semilattices with Bottom

The motivating example for this paper comes from my first paper on the variants of the convex powerset monad [1]. I removed the probabilistic content to have a simpler example in last section, but here is the full example. It can be safely skipped and it assumes you have a read [1].

Let $M = \mathcal{C}(\cdot + \mathbf{1})$ be the monad of non-empty finitely generated convex sets of subdistributions, we will show that the monad $\mathcal{C}^{\downarrow}$ can be constructed with the procedure detailed above. The main idea is that the operation of $\perp$–closure satisfies the properties of $\overline{K}$.

Definition 26. Let X be a set and let $S \in \mathcal{C}(X+\mathbf{1})$. We say that S is $\perp$–closed if for all $\varphi \in S$,

$$\{\psi \in \mathcal{D}(X+\mathbf{1}) \mid \forall x \in X, \psi(x) \leq \varphi(x)\} \subseteq S.$$

For a set X, we define $K_X : X \to \mathcal{C}(X+\mathbf{1}) = x \mapsto cc\left(\{\delta_x, \delta_\star\}\right)$. We will first show that $\overline{K}_X = \mu_X \circ \mathcal{C}(K_X + \mathbf{1})$ is the operation of $\perp$–closure, then that $\overline{K}_X$ satisfies the properties described in the previous sections and finally detail the monad we obtain.

Lemma 27. *Let X be a set, for any $S \in \mathcal{C}(X+\mathbf{1})$, $\overline{K}_X(S)$ is the smallest $\perp$–closed set containing S.*

Proof. See Theorem 35 in [1]. $\square$

Lemma 28. *The family $K_X : X \to \mathcal{C}(X+\mathbf{1})$ is natural.*

Proof. For any $f : X \to Y$, we have

$$K_Y(f(x)) = cc\left(\left\{\delta_{f(x)}, \delta_\star\right\}\right) = \mathcal{C}(f+\mathbf{1})(cc\left(\{\delta_x, \delta_\star\}\right)) = \mathcal{C}(f+\mathbf{1})(K_X(x))).$$

$\square$

Lemma 29. *The family $\overline{K}_X : \mathcal{C}(X+\mathbf{1}) \to \mathcal{C}(X+\mathbf{1})$ satisfies the following properties:*

1. *it is natural,*
2. *each component is idempotent, and*
3. *each component is a homomorphism between the free $\mathcal{C}(\cdot+\mathbf{1})$–algebras.*

Proof.

1. This is a corollary of $K : \mathrm{id}_{\mathbf{Set}} \Rightarrow \mathcal{C}(\cdot+\mathbf{1})$ being natural as shown in the following derivation. We need to show that for any $f : X \to Y$, we have $\overline{K}_Y \circ \mathcal{C}(f+\mathbf{1}) = \mathcal{C}(f+\mathbf{1}) \circ \overline{K}_X$. This follows from the following derivation.

$$\begin{aligned}
\overline{K}_Y \circ \mathcal{C}(f+\mathbf{1}) &= \mu_Y \circ \mathcal{C}(K_Y+\mathbf{1}) \circ \mathcal{C}(f+\mathbf{1}) && \text{def of } \overline{K}_Y \\
&= \mu_Y \circ \mathcal{C}(\mathcal{C}(f+\mathbf{1})+\mathbf{1}) \circ \mathcal{C}(K_X+\mathbf{1}) && \text{nat of } K \\
&= \mathcal{C}(f+\mathbf{1}) \circ \mu_X \circ \mathcal{C}(K_X+\mathbf{1}) && \text{nat of } \mu \\
&= \mathcal{C}(f+\mathbf{1}) \circ \overline{K}_X && \text{def of } \overline{K}_X
\end{aligned}$$

2. Since $\overline{K}_X(S)$ is $\perp$–closed, it is the smallest $\perp$–closed containing itself, thus $\overline{K}_X(\overline{K}_X(S)) = \overline{K}_X(S)$.
3. This holds because $\overline{K}_X$ is the image of K_X (seen as a Kleisli morphism) under the embedding of the Kleisli category of $\mathcal{C}(\cdot+\mathbf{1})$ into $\mathrm{EM}(\mathcal{C}(\cdot+\mathbf{1}))$.

$\square$

Remark 30. Apart from the second point, the above proof is very general. Namely, it shows that starting from a natural transformation $K : \mathrm{id}_{\mathbf{C}} \Rightarrow M$ such that $\overline{K}$ is idempotent, we can derive all the previous sections. This is the third item of Theorem 24.

We find that $\mathcal{C}^{\downarrow}$ is the monad of non-empty finitely generated $\perp$–closed convex sets of subdistributions with the unit being $x \mapsto K_X(x) = \overline{K}_X \circ \eta_X$. For the multiplication, there is a slight surprise; it turns out that the multiplication of $\perp$–closed sets is already $\perp$–closed, so there is no need to apply $\perp$–closure again as in the general case.

In particular, this means the inclusion $\iota : \mathcal{C}^{\downarrow} \Rightarrow \mathcal{C}(\cdot+\mathbf{1})$ is not a monad map *only* because it does not commute with the units of the two monads.

5 Conclusion

Let (M, η, μ) be a monad on a category $\mathbf{C}$ where idempotents split. If you have a natural family of idempotent homomorphisms of free M–algebras $MX \to MX$ given in either of the following ways, then you obtain a monad M^K by splitting these idempotents.

- An idempotent natural transformation $K : \mathrm{id}_{\mathrm{Kl}(M)} \Rightarrow \mathrm{id}_{\mathrm{Kl}(M)}$.
- An natural transformation $\overline{K} : M \Rightarrow M$ such that $\overline{K}_X$ is a homomorphism and it is split.
- A natural transformation $K : \mathrm{id}_{\mathbf{C}} \Rightarrow M$ such that $\mu \circ MK$ is idempotent.

References

[1] Matteo Mio, Ralph Sarkis, and Valeria Vignudelli. Combining nondeterminism, probability, and termination: Equational and metric reasoning. In *2021 36th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, pages 1–14, 2021.