

Important Results - MATH 570

Ralph Sarkis

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1 Category Theory I

Definition 1 (Category). A category is a set of objects C_0 together with a set of morphisms $\text{Mor}_C(A, B)$ for each ordered pair of objects $A, B \in C_0$ and an associative composition operation $\text{Mor}_C(A, B) \times \text{Mor}_C(B, C) \rightarrow \text{Mor}_C(A, C)$ for each ordered triple of objects $A, B, C \in C_0$. We will denote C_1 to be the set of all morphisms of C .

Definition 2 (Covariant functor). Let C and D be categories a covariant functor $F : C \rightsquigarrow D$ is a map $C_0 \rightarrow D_0$ and a map $\text{Mor}_C(A, B) \rightarrow \text{Mor}_D(F(A), F(B))$ for each ordered pair of objects $A, B \in C_0$.

- For any composable morphisms f and g , $F(f \circ g) = F(f) \circ F(g)$.
- For any object, $A \in C_0$, $F(1_A) = 1_{F(A)}$.

Definition 3 (Contravariant functor). A contravariant functor $F : C \rightsquigarrow D$ is a map $C_0 \rightarrow D_0$ and a map $\text{Mor}_C(A, B) \rightarrow \text{Mor}_D(F(B), F(A))$ for each ordered pair of objects $A, B \in C_0$.

- For any composable morphisms f and g , $F(f \circ g) = F(g) \circ F(f)$.
- For any object, $A \in C_0$, $F(1_A) = 1_{F(A)}$.

Definition 4 (Full, faithful and essentially surjective). Let $F : C \rightsquigarrow D$ be a covariant functor.

- If the restriction $F_{A,B} : \text{Mor}_C(A, B) \rightarrow \text{Mor}_D(F(A), F(B))$ is injective for any $A, B \in C_0$, then we say F is faithful.
- If $F_{A,B}$ is surjective for any $A, B \in C_0$, then F is full.
- If for any $X \in D$, there exists $Y \in C_0$ such that $D \cong F(Y)$, then F is essentially surjective.

Definition 5 (Mor functors). Let C be a category and $A \in C_0$ one of its object. We define the covariant and contravariant Mor functors from C to **Set**.

A. The functor $\text{Mor}_C(A, -) : C \rightsquigarrow \mathbf{Set}$ sends an object $B \in C_0$ to the hom set $\text{Mor}_C(A, B)$ and a morphism $f : B \rightarrow B'$ to the function

$$\text{Mor}_C(A, f) : \text{Mor}_C(A, B) \rightarrow \text{Mor}_C(A, B') = g \mapsto f \circ g.$$

B. The functor $\text{Mor}_C(-, A) : C \rightsquigarrow \mathbf{Set}$ sends an object $B \in C_0$ to the hom set $\text{Mor}_C(B, A)$ and a morphism $f : B \rightarrow B'$ to the function

$$\text{Mor}_C(f, A) : \text{Mor}_C(B', A) \rightarrow \text{Mor}_C(B, A) = g \mapsto g \circ f.$$

Definition 6 (Natural transformation). Let $F, G : C \rightsquigarrow D$ be two covariant functors, a **natural transformation** $\phi : F \Rightarrow G$ is a map $\phi : C_0 \rightarrow D_1$ that satisfies $\phi(A) \in \text{Mor}_D(F(A), G(A))$ for all $A \in C_0$ and makes the following diagram commute for any $f \in \text{Mor}_C(A, B)$:

$$\begin{array}{ccc} F(A) & \xrightarrow{\phi(A)} & G(A) \\ F(f) \downarrow & & \downarrow G(f) \\ F(B) & \xrightarrow{\phi(B)} & G(B) \end{array}$$

For two contravariant functors, the vertical arrows are reversed. If each $\phi(A)$ is an isomorphism, then we have a natural isomorphism and we denote $F \cong G$.

Definition 7 (Adjoint functors). Let $F : C \rightsquigarrow D$ and $G : D \rightsquigarrow C$ be covariant functors. We say that (F, G) are an adjoint pair if for any $A \in C_0$ and $B \in D_0$, we have an isomorphism $\phi_{A,B} : \text{Mor}_D(FA, B) \leftrightarrow \text{Mor}_C(A, GB)$. Such that for any $f \in \text{Mor}_C(A, A')$ and $g \in \text{Mor}_D(B, B')$, the following diagram commutes.

$$\begin{array}{ccc} \text{Mor}_D(F(A), B') & \xleftarrow{\phi_{A,B'}} & \text{Mor}_C(A, G(B')) \\ g \circ (-) \uparrow & & \uparrow G(g) \circ (-) \\ \text{Mor}_D(F(A), B) & \xleftarrow{\phi_{A,B}} & \text{Mor}_C(A, G(B)) \\ (-) \circ F(f) \uparrow & & \uparrow (-) \circ f \\ \text{Mor}_D(F(A'), B) & \xleftarrow{\phi_{A',B}} & \text{Mor}_C(A', G(B)) \end{array}$$

Definition 8 (Equivalence). An equivalence of categories is a pair of covariant functors $F : C \rightsquigarrow D$ and $G : D \rightsquigarrow C$ such that $F \circ G \cong 1_D$ and $G \circ F \cong 1_C$. If the functors are contravariant, we say that the pair is an anti-equivalence of categories.

2 Modules

Definition 9 (*R*-biadditive map). Let R be a ring, A be a right R module, B be a left R -module and H be an abelian group. An R -biadditive map $f : A \times B \rightarrow H$ is a map that satisfies the following for any $a, a' \in A, b, b' \in B$ and $r \in R$.

$$\begin{aligned} f(a + a', b) &= f(a, b) + f(a', b) \\ f(a, b + b') &= f(a, b) + f(a, b') \\ f(a \cdot r, b) &= f(a, r \cdot b) \end{aligned}$$

Definition 10 (Tensor product). Letting R, A and B be as above, we define the tensor product of A and B denoted $A \otimes_R B$. Define

$$G = \bigoplus_{(a,b) \in A \times B} \mathbb{Z}(a, b),$$

and let N be the normal subgroup generated by all elements of the form

$$(a + a', b) - (a, b) - (a', b) \quad (a, b + b') - (a, b) - (a, b') \quad (a \cdot r, b) - (a, r \cdot b),$$

for all $a, a' \in A, b, b' \in B$ and $r \in R$. The tensor product is the quotient G/N and general elements of $A \otimes_R B$ are of the form $\sum_i a_i \otimes b_i$, but this representation is not unique.

Proposition 11. Let R, A and B be as above, the map $\pi : A \times B \rightarrow A \otimes_R B = (a, b) \mapsto a \otimes b$ is R -biadditive and it has the universal property that for any abelian group H and R -biadditive map $f : A \times B \rightarrow H$, there is a unique group homomorphism $g : A \otimes_R B \rightarrow H$ that makes the following diagram commute:

$$\begin{array}{ccc} A \times B & \xrightarrow{\pi} & A \otimes_R B \\ & \searrow f & \downarrow g \\ & & H \end{array}$$

Proof idea. Define g to be the only possible choice, namely

$$g = \sum_i a_i \otimes b_i \mapsto \sum_i f(a_i, b_i),$$

by passing to the free group G defined above. Let $\tilde{g}(a, b) = f(a, b)$ and extend by linearity to get a homomorphism from G to H . Find that the kernel contains N and conclude that there is a unique homomorphism $A \otimes_R B \mapsto H$ with the first isomorphism theorem. \square

Proposition 12. Let R, A and B be as above and let A' and B' be right and left modules respectively with homomorphisms $f : A \rightarrow A'$ and $g : B \rightarrow B'$. Then, there is a unique group homomorphism $f \otimes g : A \otimes_R B \rightarrow A' \otimes_R B'$ such that

$$(f \otimes g) \left(\sum_i a_i \otimes b_i \right) = \sum_i f(a_i) \otimes g(b_i).$$

Proof idea. Define $f \times g = (a, b) \mapsto f(a) \otimes g(b)$ and verify it is R -biadditive to conclude the existence and uniqueness of $f \otimes g$. \square

Corollary 13. Let R, A and B be as above but with the requirement that A is a left S -module and the action of R and S on A commute. Then, $A \otimes_R B$ is a left S -module.

Proof idea. For any $s \in S$, let $[s]$ denote the homomorphism taking a to $s \cdot a$. Using the last proposition, we obtain a group homomorphism $[s] \otimes \text{id}$. Define the action of s as the action of $[s] \otimes \text{id}$ and verify it makes $A \otimes_R B$ into a left S -module. The symmetric proof works when B is a bimodule. \square

Definition 14 (R -algebra). Let R be a ring, an R -algebra is a ring A with a homomorphism $i_A : R \rightarrow A$ such that the image of R is in the center of A . A homomorphism of R -algebras A and B is a homomorphism of rings with the restriction that the image of i_A is sent to the image of i_B .

Corollary 15. Let R be a ring and A and B be R -algebras, then $A \otimes_R B$ is also an R -algebras.

Proof. For any $s \in A$, let $[s] = a \mapsto sa$ and for any $t \in B$, let $[t] = b \mapsto tb$. For any $(s, t) \in A \times B$, we get an endomorphism of $A \otimes_R B$ denoted $[s] \otimes [t]$. From the universal property of $A \otimes_R B$, we get a unique homomorphism sending $\sum_i s_i \otimes t_i$ to $\sum_i [s_i] \otimes [t_i]$. Define multiplication by

$$\left(\sum_i s_i \otimes t_i \right) \left(\sum_j a_j \otimes b_j \right) = \left(\sum_i [s_i] \otimes [t_i] \right) \left(\sum_j a_j \otimes b_j \right) = \sum_{i,j} s_i a_j \otimes t_i b_j,$$

and check that it makes $A \otimes_R B$ into a ring. The homomorphism $i_{A \otimes_R B}$ sends r to $i_A(r) \otimes 1 = 1 \otimes i_B(r)$, check that its image is in the center of the ring. \square

Corollary 16. Let R be a ring and M be an R -module, then $R \otimes_R M \cong M$.

Proof idea. We have the map $M \rightarrow R \otimes_R M$ that sends $m \mapsto 1 \otimes m$. Construct its inverse $r \otimes m \mapsto rm$ by first defining $(r, m) \mapsto rm$ and showing it is R -biadditive. \square

Corollary 17. Let R be a commutative ring and I, J be ideals, then

$$R/I \otimes_R R/J \cong R/(I + J).$$

Proof idea. Define the map $R \rightarrow R/I \otimes_R R/J$ by $r \mapsto r \otimes 1$. Check it has kernel $(I + J)$ and use FIT to get the map $R/(I + J) \rightarrow R/I \otimes_R R/J$. Next, define the map $r \otimes s \mapsto rs$ in the other direction by using the universal property and show that they are inverses. \square

Theorem 18. Let R and S be rings and ${}_S A_R$ be a bimodule, the functors

$$A \otimes_R (-) : {}_R \mathbf{Mod} \rightsquigarrow {}_S \mathbf{Mod} \quad \text{and} \quad \text{Hom}_S(A, -) : {}_R \mathbf{Mod} \rightsquigarrow {}_S \mathbf{Mod}$$

form an adjoint pair.

Proof idea. We already checked that $A \otimes_R B$ sends left R -modules to left S -modules. Define $A \otimes_R (f)$ for $f : B \rightarrow B'$ to be $\text{id} \otimes f$. Then, check that this makes $A \otimes_R (-)$ into a covariant functor.

The left R -module structure of $\text{Hom}_S(A, B)$ when B is a left S -module is given by the action $r \cdot f = a \mapsto f(a \cdot r)$. The functor sends a module homomorphism f to the post composition $f \circ (-)$. Left to check that this is a covariant functor.

Fix a left R -module B and a left S -module C . Define the morphism

$$\phi : \text{Hom}_S(A \otimes_R B, C) \rightarrow \text{Hom}_R(B, \text{Hom}_S(A, C)).$$

It sends a morphism f to $\phi(f) = b \mapsto f(- \otimes b)$. Need to check that this $\phi(f)(b)$ is a homomorphism of S -modules for any $b \in B$, that $\phi(f)$ is a homomorphism of R -modules for any f in the L.H.S. and that ϕ is a homomorphism of groups.

We now define the inverse

$$\phi^{-1} : \text{Hom}_R(B, \text{Hom}_S(A, C)) \rightarrow \text{Hom}_S(A \otimes_R B, C).$$

Let g be in the L.H.S., we note that sending (a, b) to $g(b)(a)$ is an R -biadditive map and it yields a unique homomorphism of groups $A \otimes_R B \rightarrow C$, this map is $\phi^{-1}(g)$. We need to check $\phi^{-1}(g)$ is also a homomorphism of S -modules and that ϕ^{-1} is indeed the inverse of ϕ .

We got the desired isomorphisms of groups between the adjoint pairs, but it remains to check that they are natural. \square

Definition 19. Let k be a field, G be a group and $H < G$. If V is a left $k[H]$ -module, define $\text{Ind}_H^G V = k[G] \otimes_{k[H]} V$. As seen above, this is a functor

$$\text{Ind}_H^G : {}_{k[H]} \mathbf{Mod} \rightsquigarrow {}_{k[G]} \mathbf{Mod}.$$

If W is a left $k[G]$ module, define $\text{Res}_H^G W$ to be W but only viewed as a $k[H]$ module (forgetting the action of elements not in H). This is a functor in the opposite direction.

Corollary 20. The functors $(\text{Ind}_H^G, \text{Res}_H^G)$ form an adjoint pair.

Proof idea. This is just an application of the last theorem. Letting $R = k[H]$, $S = k[G]$, $A = k[G]$ and noting that $\text{Hom}_{k[G]}(k[G], W) \cong W$. \square

3 Complex Representations of Finite Groups

Definition 21 (Representation). Let G be a finite group, V be a finite dimensional vector space over \mathbb{C} and $\rho : G \rightarrow \text{Aut}(V)$ be a group homomorphism, (ρ, V) is called a finite representation of G .

Definition 22 (Morphism of representation). Let (ρ, V) and (τ, W) be representations of a finite group G , a linear map $T : V_1 \rightarrow V_2$ is called a morphism of ρ_1 to ρ_2 if for any $g \in G$, $\rho_2(g) \circ T = T \circ \rho_1(g)$. We will denote $\text{Hom}_G(V_1, V_2)$ to be the subspace of $\text{Hom}(V_1, V_2)$ with linear maps satisfying this property.

Proposition 23. Denote $\mathbf{Rep}(G)$ to be the category of representations we have just defined, it is equivalent to the category ${}_{\mathbb{C}[G]}\mathbf{Mod}$.

Proof idea. We define two functors that compose to the identity functor.

Given a $\mathbb{C}[G]$ -modules V , define $G \rightarrow \text{Aut}(V)$ by sending g to $\rho(g) = v \mapsto g \cdot v$ (where the product comes from the module structure). Verify that this is indeed a representation of G . Conversely, given the representation (ρ, V) , make V into a $\mathbb{C}[G]$ -module by letting G act on it via ρ and extend the action by linearity.

Since the underlying sets of the objects do not change under the functors, morphisms are not modified either. However, we still need to verify that they are homomorphism in the target category. \square

Definition 24 (Character group). For a group G , the character group of G , denoted G^* , is the set of group homomorphisms from G to \mathbb{C}^\times .

Proposition 25. The following are properties of the character group.

1. $(H \times G)^* \cong H^* \times G^*$.
2. $(\mathbb{Z}/n\mathbb{Z})^* \cong \mathbb{Z}/n\mathbb{Z}$.
3. If G is finite and abelian, $G^* \cong G$.
4. For a general group G , $G^* = (G/G')^*$.

Proof idea.

1. Define the map $\phi : (H \times G)^* \rightarrow H^* \times G^*$ with $f \mapsto (f(\cdot, 1), f(1, \cdot))$. Show that it is an isomorphism.
2. Observe that $f \in (\mathbb{Z}/n\mathbb{Z})^*$ is only defined by where it sends the generator, and it must send it to a generator of the group of n th roots of unity (this group is isomorphic to $\mathbb{Z}/n\mathbb{Z}$).
3. Use the structure theorem and the two previous points.
4. Show that if $f \in G^*$, then $f([x, y]) = 1$ and so $G' \subseteq \ker(f)$, then the result follows from the first isomorphism theorem. \square

Theorem 26. Let (ρ, V) be a representation of G , there exists a inner product that is G -invariant (i.e. for all $v, w \in V$, $\langle \rho(g)v, \rho(g)w \rangle = \langle v, w \rangle$).

Proof idea. Take any inner product $\langle \cdot, \cdot \rangle$ and let $\langle u, v \rangle = \frac{1}{|G|} \sum_{g \in G} \langle \rho(g)u, \rho(g)v \rangle$, verify that $\langle \cdot, \cdot \rangle$ is G -invariant. \square

Theorem 27. Any representation decomposes as a sum of irreducible representations.

Proof idea. Argue by induction. If U is a subrepresentation, then U^\perp (w.r.t. a G -invariant inner product) is also a subrepresentation. \square

Theorem 28. Let G be an abelian group, every representation of G decomposes into a direct sum of 1-dimensional representations.

Proof idea. First prove that $\rho(g)$ is diagonalizable. Then use the fact that commuting diagonalizable linear operator are simultaneously diagonalizable. \square

Lemma 29 (Schur). Let (ρ, V) and (τ, W) be irreducible representations of G , we have the following:

$$\text{Hom}_G(V, W) \cong \begin{cases} 0 & \rho \not\cong \tau \\ \mathbb{C} & \rho \cong \tau \end{cases}$$

Proof idea. Note that if $T \in \text{Hom}_G(V, W)$, $\ker(T)$ and $\text{Im}(T)$ are subrepresentations, this implies T is either trivial or an isomorphism. Now, look at an eigenspace of T and show that it must be equal to the whole vector space. \square

Definition 30. Let (ρ, V) and (τ, W) be representations of G , $\sigma : G \rightarrow \text{Aut}(\text{Hom}(V, W))$ is a new representation with $\sigma(g)T = \tau(g) \circ T \circ \rho(g^{-1})$.

Theorem 31. We get that for any $g \in G$, $\chi_\sigma(g) = \overline{\chi_\rho(g)} \chi_\tau(g)$.

Proof idea. No need to learn it. \square

Definition 32. Let (ρ, V) be a representation of G , define the projection operator as $\pi_\rho : V \rightarrow V$ with $\pi_\rho = \frac{1}{|G|} \sum_{g \in G} \rho(g)$.

Theorem 33. If $\rho = \rho_1^{a_1} \oplus \dots \oplus \rho_t^{a_t}$ where ρ_1 is the trivial representation, then

$$\pi_\rho = \text{Id}_{V_1^{a_1}} \oplus 0 \oplus \dots \oplus 0$$

From this, we get the following:

$$a_1 = \text{Tr}(\pi_\rho) = \frac{1}{|G|} \sum_{g \in G} \chi_\rho(g) = \langle \chi_\rho, \chi_1 \rangle$$

Proof idea. Note that $V^G = (V_1^{a_1})^G \oplus \dots \oplus (V_1^{a_t})^G$ and that except for $i = 1$, $(V_i^{a_i})^G = \{0\}$ because it is a subrepresentation. The result follows. \square

Theorem 34. *The characters of irreducible representations are orthogonal with respect to the G -invariant inner product.*

Proof idea. Use $\dim(\text{Hom}(V, W)^G) = \frac{1}{|G|} \sum_{g \in G} \chi_\sigma(g) = \langle \chi_\rho, \chi_\tau \rangle$. Then use Schur's lemma. \square

Proposition 35. *Here are some consequences of the last theorem.*

1. A representation ρ decomposes into an irreducible representation: $\rho = \rho_1^{a_1} \oplus \dots \oplus \rho_t^{a_t}$.
2. $a_i = \langle \chi_\rho, \chi_{\rho_i} \rangle$.
3. χ_ρ determines ρ up to isomorphism.
4. $\rho^{\text{reg}} = \rho_1^{\dim(\rho_1)} \oplus \dots \oplus \rho_t^{\dim(\rho_t)}$.
5. ρ is irreducible if and only if $\|\chi_\rho\| = 1$.
6. There exists finitely many irreducible characters (hence representations).

Proof idea.

1. Done above.
2. Follows from orthogonality of the irreducible characters.
3. Follows from the last part.
4. Follows from the fact that χ_{reg} is 0 everywhere but on the identity. Also implies $|G| = \sum_i \dim(\rho_i)^2$, where the ρ_i 's are the irreducible representations.
5. Follows from orthonormality of the irreducible characters.
6. Since they are orthonormal, there cannot be more than the dimension of $\text{Class}(G)$. \square

Definition 36. We define a more general operator. Let $\alpha \in \text{Class}(G)$, we define the operator $A_\rho = \sum_{g \in G} \alpha(g)\rho(g)$.

Lemma 37. *For two representations ρ and τ of G , $A_{\rho \oplus \tau} = A_\rho \oplus A_\tau$.*

Proof idea. Use the definitions. \square

Theorem 38. *Let $\chi_{\rho_1}, \dots, \chi_{\rho_t}$ be the characters of all the irreducible representations of G , they form an orthonormal basis of $\text{Class}(G)$, in particular, $t = h(G)$.*

Proof idea. Let $\beta \in \text{Class}(G)$ be a function orthogonal to all irreducible characters. Let $\alpha = \bar{\beta}$ and for an irreducible representation ρ_i , show that $A_{\rho_i} \equiv 0$. Using the last lemma, we get $A_{\rho^{\text{reg}}} \equiv 0$, which is equivalent to $\alpha = \bar{\beta} \equiv 0$. \square

Theorem 39. Let $H < G$ be groups and (ρ, V) be a representation of H . The induced representation is constructed by taking $\text{Ind}_H^G V$ as a $\mathbb{C}[G]$ -module and looking at the corresponding representation. Its character can be calculated like so:

$$\text{Ind}_H^G \chi(g) = \frac{1}{|H|} \sum_{\{b \in G | b^{-1}gb \in H\}} \chi(b^{-1}gb).$$

Proof idea. Writing $G = \coprod_{i=1}^d g_i H$ yields the decomposition $\text{Ind}_H^G V \cong \bigoplus_{i=1}^d g_i \otimes V$. Observe how G acts on this set and conclude that $\rho(g)$ is a matrix with d square blocks of size $\dim(V)$. Only the diagonal blocks contribute to the matrix and they occur when $gg_i = g_i h$ for some h . We can conclude the formula by not only considering g_i 's and averaging over the size of H . \square

Corollary 40. If $H \triangleleft G$, then the formula can be written as

$$\text{Ind}_H^G \chi(g) = \begin{cases} 0 & g \notin H \\ \frac{1}{|H|} \sum_{b \in G} \chi(b^{-1}gb) & g \in H \end{cases}.$$

Theorem 41 (Frobenius reciprocity). Let $H < G$ be groups and $\langle \cdot, \cdot \rangle_X$ denote the inner product of the class functions on $X \in \{H, G\}$. Let ρ be a representation of H and τ a representation of G , we have

$$\langle \text{Ind}_H^G \rho, \tau \rangle_G = \langle \rho, \text{Res}_H^G \tau \rangle_H.$$

Proof idea. This is just a straightforward calculation with rearrangements in the sum. \square

Corollary 42. Let σ be an irreducible representation of $H < G$. The representation $\text{Ind}_H^G \sigma$ is irreducible if and only if for all $g \in G \setminus H$, $\sigma^g \not\cong \sigma$ where $\sigma^g = h \mapsto \sigma(g^{-1}hg)$.

Proof idea. Calculate the norm of $\text{Ind}_H^G \chi_\sigma$ using Frobenius reciprocity to get $\frac{1}{|H|} \sum_{g \in G} \langle \chi_\sigma, \chi_{\sigma^g} \rangle$. Next, show that σ^g is always irreducible for $g \in G \setminus H$, to remove terms of the sum. The equivalence is now clear to see. \square

Definition 43. A group G is supersolvable if it has a composition series where each group is normal in G and each quotient is cyclic.

Theorem 44 (Blichfeldt). Let G be a supersolvable group and ρ an irreducible representation of G , then there exists a subgroup $J < G$ and one-dimensional representation ψ of J such that $\rho \cong \text{Ind}_J^G \psi$.

Definition 45 (Graded ring). A ring R is called graded if it decomposes as a direct sum of abelian groups $\{R_n \mid n \in \mathbb{N}\}$ such that for any m, n , $R_m R_n \subseteq R_{n+m}$.

Definition 46 (Tensor algebra). Let R be a commutative ring and V an R -module. Denote $T^0(V) = R$, $T^1(V) = V$ and generally $T^n(V) = \bigotimes_R^n V$ and finally define $T^\bullet(V) = \bigoplus_{n \in \mathbb{N}} T^n(V)$ to be the tensor algebra of V . This is a graded R -algebra.

Definition 47 (Symmetric algebra). In the same setting as above, let $I_0 = I_1 = 0$, let I_2 be the R -span of the elements $\{x \otimes y - y \otimes x \mid x, y \in V\}$ and generally let I_n be the R -span of

$$\{x_1 \otimes \cdots \otimes x_n - x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)} \mid \forall i, x_i \in V, \sigma \in S_n\}.$$

Define the symmetric algebra to be $\text{Sym}^\bullet(V) = \bigoplus_{n \in \mathbb{N}} T^n(V)/I_n$ and $\text{Sym}^n(V) = T^n(V)/I_n$.

Definition 48 (Exterior algebra). In the setting of above, let $J_0 = J_1 = 0$ and let J_n be the R -span of the elements

$$\{x_1 \otimes \cdots \otimes x_n \mid \forall i, x_i \in V, \exists i < j, x_i = x_j\}.$$

Define the exterior algebra to be $\bigwedge^n V = \bigoplus_{n \in \mathbb{N}} T^n(V)/J_n$ and $\bigwedge^n V = T^n(V)/J_n$.

Proposition 49. In the setting of above, suppose V is a free module of rank V , then

$$\dim(\text{Sym}^n(V)) = \binom{n+d-1}{n} \quad \text{and} \quad \dim(\bigwedge^n V) = \binom{d}{n}.$$

Proposition 50. Let V be a representation of G with character χ , the representation $T^2(V)$ and $\text{Sym}^2(V)$ have characters χ^2 and $\frac{1}{2}(\chi^2 + \chi(-^2))$.

Proposition 51. Let $\rho^{\text{St},0}$ be the non-trivial irreducible representation of S_n contained in ρ^{St} , then for any $1 \leq a \leq n-1$, $\bigwedge^a \rho^{\text{St},0}$ is irreducible.

Proposition 52. Let R be a commutative ring, V and W be R -modules, then there exists R -module isomorphisms between the R - d -multilinear maps $V \rightarrow W$ and homomorphisms $T^d(V) \rightarrow W$, between R - d -multilinear symmetric maps $V \rightarrow W$ and homomorphisms $\text{Sym}^d(V) \rightarrow W$ and between R - d -multilinear antisymmetric maps $V \rightarrow W$ and homomorphisms $\bigwedge^d V \rightarrow W$.

4 Representations of S_n

Definition 53 (Young). Let λ be a partition ($\lambda_1 \geq \cdots \geq \lambda_r$) of n . A Young diagram associated to λ consists of r aligned rows of boxes where the i -th row has λ_i boxes. A Young tableau is a Young diagram filled with integers in $\{1, \dots, n\}$. The conjugate partition of λ is the partition corresponding to the number of boxes in each column. The diagram corresponding to the conjugate partition is the original diagram flipped along the $y = -x$ line. The standard tableau is when the boxes are filled in order going from left to right and top to bottom.

Definition 54 (Young symmetrizer). Let λ be a partition of n , we define P_λ and Q_λ considering the standard tableau associated to λ :

$$P_\lambda = \{\sigma \in S_n \mid \sigma \text{ preserves every row of the tableau}\}$$

$$Q_\lambda = \{\sigma \in S_n \mid \sigma \text{ preserves every column of the tableau}\}.$$

The Young symmetrizer of λ is denoted $c_\lambda = a_\lambda b_\lambda$ where a_λ and b_λ are elements of the group ring $\mathbb{C}[S_n]$ defined by

$$a_\lambda = \sum_{\sigma \in P_\lambda} \sigma \quad b_\lambda = \sum_{\sigma \in Q_\lambda} \sigma.$$

Lemma 55. *Let λ be a partition of n , then the following holds:*

1. *For any $p \in P_\lambda$, $a_\lambda p = pa_\lambda = a_\lambda$ and $pc_\lambda = c_\lambda$.*
2. *For any $q \in Q_\lambda$, $qb_\lambda = b_\lambda q = \text{sgn}(q)b$ and $c_\lambda q = \text{sgn}(q)c_\lambda$.*
3. *For any $x \in \mathbb{C}[S_n]$ such that for any $p \in P_\lambda$ and $q \in Q_\lambda$, $pxq = \text{sgn}(q)x$, we have $x = kc_\lambda$ with $k \in \mathbb{C}$.*

Lemma 56. *Let T be a Young tableau and $g \in S_n$. Denote gT to be the tableau obtained after multiplying every element in T by g on the left. Suppose no pair of distinct integers appear in the same row of T and the same column of gT , then $g \in PQ$.*

Corollary 57. *For any $z \in \mathbb{C}[S_n]$, we have $c_\lambda z c_\lambda \in \mathbb{C}c_\lambda$. In particular, $c_\lambda^2 = n_\lambda c_\lambda$ for some $n_\lambda \in \mathbb{C}$.*

Lemma 58. *Let G be a finite group and W be isomorphic to a subrepresentation of $\mathbb{C}[G]$. There exists $\phi \in \mathbb{C}[G]$ such that $W \cong \mathbb{C}[G]\phi$ and $\phi^2 = \phi$.*

Theorem 59. *For any partition λ of n , $V_\lambda = \mathbb{C}[S_n]c_\lambda$ is an irreducible representation.*

Proposition 60. *Let λ be a partition of n , then $n_\lambda = \frac{n!}{\dim(V_\lambda)}$.*

Lemma 61. *If $\lambda < \mu$ in the lexicographic order of partition on n , then $c_\lambda \mathbb{C}[G]c_\mu = 0$ and $c_\lambda c_\mu = 0$.*

Theorem 62. *Let $\lambda \neq \mu$ be partitions of n , then $V_\lambda \not\cong V_\mu$.*

Corollary 63. *All irreducible representations of S_n are isomorphic to V_λ for some partition λ of n .*

Definition 64 (Hook length). In a Young diagram, a hook associated to a box is the set of boxes either below or to the right of this box (including the initial box). The hook length is the size of this set.

Proposition 65. *Let λ be a partition of n , then*

$$\dim(V_\lambda) = \frac{n!}{\prod_{h \text{ is a hook}} (\text{hook length of } h)}.$$

5 Category Theory II

Definition 66 (Hom functors). Let C be a category and A one of its objects. We associate to A a covariant functor $h_A = \text{Mor}_C(A, -) : C \rightsquigarrow \mathbf{Sets}$ and a contravariant functor $h^A = \text{Mor}_C(-, A) : C \rightsquigarrow \mathbf{Sets}$.

Lemma 67 (Yoneda). Let $F : C \rightsquigarrow \mathbf{Sets}$ be a covariant functor and $\text{Nat}(h_A, F)$ be the set of natural transformations from h_A to F . There is a natural bijection $\text{Nat}(h_A, F) \leftrightarrow F(A)$ given by

$$\begin{aligned} \phi &\mapsto \phi_A(id_A) & \forall \phi \in \text{Nat}(h_A, F) \\ u &\mapsto \{\phi_B = f \mapsto F(f)(u) \mid \forall B \in C_0\} & \forall u \in F(A). \end{aligned}$$

Corollary 68. For any objects A, B of a category C , $h_A \cong h_B$ if and only if $A \cong B$.

Definition 69 (Representable functor). A functor $C \rightsquigarrow \mathbf{Sets}$ is called representable if there exists an object $X \in C$ such that this functor is naturally isomorphic to $\text{Mor}_C(X, -)$ and covariant or it is naturally isomorphic to $\text{Mor}_C(-, X)$ and contravariant.

Theorem 70. Let $F : C \rightsquigarrow D$ be a covariant functor, there exists a functor $G : D \rightsquigarrow C$ such that (F, G) is an equivalence of categories if and only if F is fully faithful and essentially surjective.

Theorem 71 (Morita). Let R be a ring and $n \in \mathbb{N}$ be non-zero. The functor $F : {}_R\mathbf{Mod} \rightsquigarrow M_n(R)\mathbf{Mod}$ defined by sending a module M to M^n and a morphism f to $f^n = (f, \dots, f)^t$ is an equivalence of categories.

Definition 72 (Division algebra). A ring R is called a division ring if every non-zero element is a unit. The center of R is a field, so we also say that R is a division algebra over this field.

Proposition 73. Any division algebra over \mathbb{R} is either \mathbb{R} , \mathbb{C} or \mathbb{H} , the Hamilton quaternions.

Theorem 74. Any finite division ring is a field.

Definition 75 (Universal property). ¹ Let C be a category, X an object and J a commutative diagram in C . We say that X satisfies an initial property if for any object Y that can fit in the diagram J in the place of X , there is a unique morphism $f : X \rightarrow Y$ such that we can fit both X and Y in the diagram J , add the morphism f and obtain a commutative diagram. We say that X satisfies a terminal property if the direction of the unique morphism is reversed.

An object satisfies a universal property if it satisfies a initial property or a terminal property.

¹This is a rough and not too formal definition, but encapsulates what is needed to know in this course.

Definition 76 (Directed/injective system). Let C be a category, I be a poset, $\{X_i \mid i \in I\}$ be a set of objects of C and $\{f_{ij} \mid \forall i \leq j\}$ be such that $f_{ij} \in \text{Mor}_C(X_i, X_j)$, $f_{ii} = \text{id}_{X_i}$ and $f_{jk} \circ f_{ij} = f_{ik}$ for any $i \leq j \leq k \in I$. It can also be defined as a covariant functor from the category representing the poset I to C .

Definition 77 (Cocone). Let I be a poset and $F : I \rightsquigarrow C$ be a directed system. A cocone of $F(I)$ is an object X along with morphisms $e_i : F(i) \rightarrow X$ such that for any $i \leq j \in I$, $e_j \circ f_{ij} = e_i$.

Let $X, \{e_i\}$ and $Y, \{d_i\}$ be cocones of the directed system. A morphism of cocones is a morphism $m : X \rightarrow Y$ such that for any $i \in I$, $m \circ e_i = d_i$. This yields the category of cocones of the directed system.

Definition 78 (Direct limit). Given a directed system as above, the direct limit of this system, if it exists, is an object $X \in C$ and morphisms $e_i : X_i \rightarrow X$ that form an initial object in the category of cocones of this system. We denote the limit $\varinjlim X_i = X$.

Proposition 79. *In the category of **Sets**, the direct limit of any directed system $\{X_i\}, \{f_{ij}\}$ exists.*

Proof idea. Let S be the disjoint union of the X_i 's, formally $S = \coprod_{i \in I} \{i\} \times X_i$ and \sim be the equivalence relation given by the symmetric extension of $(i, x) \sim (j, f_{ij}(x))$ for all $i \leq j \in I$. We claim that $\varinjlim X_i = S / \sim$ with morphisms $e_i : X_i \rightarrow S / \sim$ defined by sending x to $[(i, x)]$. Left to check this is an initial cocone. \square

Definition 80 (Direct sum/Coproduct). Let I be a discrete poset (no distinct elements are comparable) and $\{X_i\}$ form a directed system. The directed limit of this system is called the direct sum or coproduct of the X_i 's. It is often denoted $\bigoplus_{i \in I} X_i$ or $\coprod_{i \in I} X_i$.

Proposition 81. *In the category of $R\text{Mod}$ for a ring R , direct sums exist for any set of objects $\{X_i\}$ ex.*

Proof idea. Define $X = \bigoplus_{i \in I} X_i$ as the set of functions from I into the disjoint union of the X_i 's such that $f(i) \in X_i$ and for all but finitely many $i \in I$, $f(i) = 0_{X_i}$. Check that this is a module and that with the morphisms $e_i : X_i \rightarrow X$ given by $x \mapsto a$ with $a_i = x$ and $a_j = 0$ for $j \neq i$, it forms an initial cocone. \square

Proposition 82. *In the category of $R\text{Mod}$, the direct limit of any directed system $\{X_i\}, \{f_{ij}\}$ exists.*

Proof idea. Let the e_i 's be defined as above, $W \subseteq \bigoplus_{i \in I} X_i$ be the R -module generated by all elements of the form $e_i(x) - e_j(f_{ij}(x))$ and let $L = (\bigoplus_{i \in I} X_i) / W$. The maps $d_i : X_i \rightarrow L$ are defined by $d_i = \pi \circ e_i$ where π is the quotient map from $\bigoplus_{i \in I} X_i$ to L . Left to show this yields an initial cocone. \square

Proposition 83. *The directed system $F : \mathbb{N} \rightarrow \text{FiniteSets}$ with $F(i) = \{1, \dots, i\}$ and $F(i \leq j)$ being the inclusion map has no direct limit in **FiniteSets**.*

Definition 84 (Pushout). Let C be a category with morphism $f : A \rightarrow B$ and $g : A \rightarrow C$. We call the direct limit of the following diagram (where identity morphisms are omitted) the pushout of the diagram.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ g \downarrow & & \\ & & C \end{array}$$

Corollary 85. *Pushouts exist in \mathbf{Sets} and ${}_R\mathbf{Mod}$ for a ring R .*

Proposition 86 (Amalgamated product). *In \mathbf{Grp} , the pushout of the diagram as above exists and it is called the amalgamated product and is denoted $B *_A C$.*

Proof idea. The underlying set is the set of all finite length words in the disjoint union of B and C where we mod the equivalence relation generated by the following requirements:

1. If for some i , x_i and x_{i+1} are in the same group and $y = x_i \cdot x_{i+1}$, then $x_1 \cdots x_n \sim x_1 \cdots x_{i-1} y x_{i+1} \cdots x_n$.
2. If for some i , $x_i = 1$, then $x_1 \cdots x_n \sim x_1 \cdots x_{i-1} x_{i+1} \cdots x_n$.
3. For every i and $a \in A$, if $x_i \in B$, $y = x_i f(a)$ and $z = g^{-1}(a) x_{i+1}$, then $x_1 \cdots x_n \sim x_1 \cdots x_{i-1} y z x_{i+1} \cdots x_n$.
4. For every i and $a \in A$, if $x_i \in C$, $y = x_i g(a)$ and $z = f^{-1}(a) x_{i+1}$, then $x_1 \cdots x_n \sim x_1 \cdots x_{i-1} y z x_{i+1} \cdots x_n$.

The multiplication is word concatenation. The maps $e_X : X \rightarrow B *_A C$ are defined for $X \in \{A, B, C\}$ by $a \mapsto f(a) \sim g(a)$, $b \mapsto b$ and $c \mapsto c$ respectively. Left to check this is the pushout.

Observe that when $A = \{1\}$ this corresponds to the free product. □

Definition 87 (Projective/inverse system). Let I be a poset and C a category, a projective system is a contravariant functor $F : I \rightsquigarrow C$.

Definition 88 (Cone). Let I be a poset and $F : I \rightsquigarrow C$ be an inverse system. A cone of $F(I)$ is an object X along with morphisms $e_i : X \rightarrow F(i)$ such that for any $i \leq j \in I$, $f_{ij} \circ e_j = e_i$.

Let $X, \{e_i\}$ and $Y, \{d_i\}$ be cones of the inverse system. A morphism of cones is a morphism $m : X \rightarrow Y$ such that for any $i \in I$, $e_i \circ m = d_i$. This yields the category of cones of the inverse system.

Definition 89 (Inverse limit). Given an inverse system as above, the inverse limit of this system, if it exists, is an object $X \in C$ and morphisms $e_i : X \rightarrow X_i$ that form a terminal object in the category of cones of this system. We denote the limit $\varprojlim X_i = X$.

Definition 90. The inverse limits for discrete systems are called direct products, they are denoted $\prod_{i \in I} X_i$.

Proposition 91. *Direct products of any sets $\{X_i\}$ exist in **Sets**.*

Proof idea. The elements of $\prod_{i \in I} X_i$ are functions $f : I \rightarrow \prod_{i \in I} X_i$ such that $f(i) \in X_i$. With the projection functions $p_i : \prod_{i \in I} X_i \rightarrow X = f \mapsto f(i)$, we obtain a terminal cone of the system. \square

Proposition 92. *The inverse limit of any inverse system $\{X_i\}, \{f_{ij}\}$ exists in **Sets**.*

Proof idea. Let X be the set of elements $f \in \prod_{i \in I} X_i$ that satisfy $f(i) = f_{ij}(f(j))$ for every $i \leq j$. With the restrictions of the projection maps to X , we get a terminal cone of the system. \square

Proposition 93. *The inverse limit of any inverse system $\{R_i\}, \{f_{ij}\}$ exists in **Rings**.*

Proof idea. The underlying set and the projection maps are the same as for the inverse limits in **Sets**, we just need to check that $\prod_{i \in I} R_i$ is a ring under coordinate-wise addition and multiplication. \square

Proposition 94. *The inverse limit of any inverse system $\{M_i\}, \{f_{ij}\}$ exists in ${}_R\mathbf{Mod}$ for any ring R .*

Proof idea. The underlying set and the projection maps are the same as for the inverse limits in **Sets**, we just need to check that $\prod_{i \in I} M_i$ is a module under coordinate-wise addition and the action $r \cdot f = f(r \cdot -)$. \square

Definition 95 (Pullback). Let C be a category with morphism $f : A \rightarrow C$ and $g : B \rightarrow C$. We call the inverse limit of the following diagram (where identity morphisms are omitted) the pullback of the diagram.

$$\begin{array}{ccc} & & A \\ & & \downarrow f \\ B & \xrightarrow{g} & C \end{array}$$

Definition 96 (Completion of a ring). Let R be a ring and $I \triangleleft R$ and denote I^n to be the ideal generated by all products of n elements of I . We have an inverse system indexed by \mathbb{N} with $F(n) = R/I^n$ and $F(n \leq m)$ being the canonical quotient map $R/I^m \rightarrow R/I^n$. The completion of the ring R relative to I is the inverse limit of this system, it is denoted $R^{\wedge I}$ and has the following underlying set:

$$\{(\dots, r_{n+1}, r_n, \dots, r_1) \mid \forall n \in \mathbb{N}, r_n \in R/I^n, r_{n+1} \equiv r_n \pmod{I^n}\}.$$

We also have a natural map $R \rightarrow R^{\wedge I}$ given by $r \mapsto (\dots, r, r, r)$.

Theorem 97 (Krull). Let R be a Noetherian domain and $I \neq R$ be an ideal of R . The natural map $R \rightarrow R^{\wedge I}$ is injective.

Definition 98 (Power series). Let A be a commutative ring and $R = A[x_1, \dots, x_n]$ be the ring of polynomials in n variables over A and let $I = \langle x_1, \dots, x_n \rangle$, then $R^{\wedge I} = A[[x_1, \dots, x_n]]$, the ring of power series of n variables over A .

Definition 99 (p -adic). Let p be prime, the ring of p -adic numbers denoted \mathbb{Z}_p is the completion of \mathbb{Z} relative to $p\mathbb{Z}$.

6 Commutative Algebra

In the following R denotes a commutative ring.

Definition 100 (Multiplicative set). We say that a subset $S \subseteq R$ is multiplicative if $1 \in S$ and S is closed under multiplication.

Definition 101 (Localization of rings). Let S be a multiplicative set in R . The localization of R at S^{-1} is $R[S^{-1}]$ is a ring where all the elements of S are invertible. Let $R_S = \{\frac{r}{s} \mid r \in R, s \in S\}$ and \sim be the equivalence relation with $\frac{r}{s} \sim \frac{r'}{s'}$ for all $r, r' \in R, s, s' \in S$ such that there exists $s'' \in S$ with $s''(s'r - sr') = 0$. We let $R[S^{-1}] = R_S / \sim$ and addition and multiplication be defined as usual for fractions, this yields a commutative ring.

Proposition 102. Let $\ell : R \rightarrow R[S^{-1}] = r \mapsto \frac{r}{1}$. This is a homomorphism with the universal property that for any homomorphism $f : R \rightarrow B$ into a commutative ring with $f(S) \subseteq B^\times$, there is a unique map $F : R[S^{-1}] \rightarrow B$ such that $f = F \circ \ell$.

Definition 103 (Localization of modules). Let M be an R -module and S a multiplicative set of R . The localization of M at S is a $R[S^{-1}]$ -module given by $M[S^{-1}] = R[S^{-1}] \otimes_R M$.

Proposition 104. Let $M_S = \{\frac{m}{s} \mid m \in M, s \in S\}$ and \sim be the equivalence relation with $\frac{m}{s} \sim \frac{m'}{s'}$ for all $m, m' \in M, s, s' \in S$ such that there exists $s'' \in S$ with $s''(s'm - sm') = 0$. Putting the obvious module structure on M_S / \sim we get a $R[S^{-1}]$ module isomorphic to $M[S^{-1}]$.

Definition 105 (Localization at an ideal). Let \mathfrak{p} a prime ideal of R , then $R - \mathfrak{p}$ is a multiplicative set and we denote $R_{\mathfrak{p}} = R[(R - \mathfrak{p})^{-1}]$ and $M_{\mathfrak{p}} = M[(R - \mathfrak{p})^{-1}]$.

Definition 106 (SES). A short exact sequence is a sequence of modules and morphisms

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

such that the image of every map is the kernel of the following one.

Definition 107 (Exact functor). Let $F : C \rightarrow D$ be a covariant additive functor between additive full subcategories of ${}_R\mathbf{Mod}$ closed under taking kernels and quotients. We say that F is exact if the image under F of a SES in C is an SES in D .

Proposition 108. *Localization is an exact functor.*

Proposition 109. *Let S be a multiplicative set of R , there is a bijection between the prime ideals of $R[S^{-1}]$ and the prime ideals of R that do not intersect S .*

Proof idea. Recall the map $\ell : R \rightarrow R[S^{-1}]$. For any $I \triangleleft R[S^{-1}]$, denote $I^c = \ell^{-1}(I)$, this is an ideal of R and if I is prime, then I^c is a prime ideal not intersecting S . For any $I \triangleleft R$, denote $I^e = I[S^{-1}]$, this is an ideal of $R[S^{-1}]$ and if I is prime and does not intersect S , then I^e is prime or $I^e = R[S^{-1}]$. Finally, we can show $I^{ec} = I$ and $I^{ce} = I$ which yields the desired bijection and also implies I^e is a proper ideal. \square

Definition 110 (Local ring). We say that R is local if it has a unique maximal ideal.

Corollary 111. *Let \mathfrak{p} be a prime ideal of R , then $R_{\mathfrak{p}}$ is a local ring.*

Definition 112 (Spectrum). The spectrum of R is $\text{Spec}(R) = \{[\mathfrak{p}] \mid \mathfrak{p} \text{ a prime ideal of } R\}$.

Lemma 113. *A homomorphism of rings $f : R \rightarrow S$ induces a function $f^* : \text{Spec}(S) \rightarrow \text{Spec}(R) = [\mathfrak{p}] \mapsto [f^{-1}(\mathfrak{p})]$.*

Definition 114 (Radical). Let I be an ideal of R , its radical is denoted \sqrt{I} and it is the ideal $\{r \in R \mid \exists n \in \mathbb{N}, r^n \in I\}$. We say that I is radical if $I = \sqrt{I}$. It is clear that a prime ideal is radical and the radical of any ideal is radical.

Definition 115 (Topology on $\text{Spec}(R)$). For any ideal I of R , define $V(I) = \{[\mathfrak{p}] \in \text{Spec}(R) \mid I \subseteq \mathfrak{p}\}$. We assign a topology on $\text{Spec}(R)$ by letting $\{V(I) \mid I \triangleleft R\}$ be the set of all closed sets.

Proposition 116. *The object defined above is indeed a topology. Namely,*

1. *There exists ideals I and J such that $\emptyset = V(I)$ and $\text{Spec}(R) = V(J)$.*
2. *For any ideals I and J , $V(I) \cup V(J) = V(IJ)$ is closed.*
3. *For any ideals $\{I_\alpha\}$, $\bigcap_\alpha V(I_\alpha) = V(\sum_\alpha I_\alpha)$ is closed.*

Lemma 117. *Let I be an ideal of R , then $\sqrt{I} = \bigcap \{\mathfrak{p} \mid I \subseteq \mathfrak{p} \text{ is prime}\}$.*

Corollary 118. *For any ideals $I, J \triangleleft R$, $V(I) = V(J)$ if and only if $\sqrt{I} = \sqrt{J}$.*

Proposition 119. *If \mathfrak{p} is a prime ideal, then $V(\mathfrak{p})$ is the closure of $\{[\mathfrak{p}]\}$.*

Proposition 120. *Let $f : R \rightarrow S$ be a ring homomorphism, the induced map f^* is continuous in the topology we defined.*

Proof idea. Let I be an ideal of R , we claim that $(f^*)^{-1}(V(I)) = V(\langle f(I) \rangle)$. \mathfrak{p} is a prime ideal containing $f(I) \Leftrightarrow f^{-1}(\mathfrak{p})$ contains $I \Leftrightarrow f^*[\mathfrak{p}] \in V(I) \Leftrightarrow \mathfrak{p} \in (f^*)^{-1}(V(I))$. \square

Proposition 121. Let $f \in R$, we denote $D(f) = \{[\mathfrak{p}] \mid f \notin \mathfrak{p}\} = \text{Spec}(R) - V(f)$.

Proposition 122. The function $\ell^* : \text{Spec}(R[\langle f \rangle^{-1}]) \rightarrow \text{Spec}(R)$ is a homeomorphism onto $D(f)$.

Definition 123. Let X be a topological space, we define the category T_X where the objects are open sets and for any open sets U, V $\text{Mor}_{T_X}(U, V) = \{i_{U,V}\}$ if $U \subseteq V$ and $\text{Mor}_{T_X}(U, U) = \emptyset$ otherwise.

Definition 124 (Sheaf). Let X be a topological space, a sheaf \mathcal{O} of commutative rings on X is a contravariant functor $\mathcal{O} : T_X \rightsquigarrow \mathbf{Rings}$ (we denote $\mathcal{O}(i_{U,V}) = \text{res}_{VU}$ or $|_U$) such that the following holds:

1. $\mathcal{O}(\emptyset) = 0$.
2. If $U = \cup_i U_i$ and $s \in \mathcal{O}(U)$ is such that $s|_{U_i} = 0$ for all i , then $s = 0$.
3. If $U = \cup U_i$ and for all i , $s_i \in \mathcal{O}(U_i)$ are elements such that for all i, j , $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, then there exists an element $s \in \mathcal{O}(U)$ such that $s|_{U_i} = s_i$ for all i .

Definition 125 (Ringed space). A ringed space is a topological space X along with a sheaf of rings \mathcal{O}_X .

Definition 126 (Stalk). Given $x \in X$ a point in a ringed space, the stalk or germs of functions at x is the ring $\mathcal{O}_{X,x} = \varinjlim \{\mathcal{O}_X(U) \mid x \in U\}$. Elements of this ring are pairs (U, f) where U is an open set containing x and $f \in \mathcal{O}_X(U)$ under the equivalence relation $(U, f) \sim (V, g)$ whenever $f|_{U \cap V} = g|_{U \cap V}$. Addition and multiplication is defined by

$$(U_1, f_1) \square (U_2, f_2) = (U_1 \cap U_2, f_1|_{U_1 \cap U_2} \square f_2|_{U_1 \cap U_2}).$$

Definition 127 (Locally ringed space). A locally ringed space is a ringed space such that the stalk at any point is a local ring.

Proposition 128. We can make $\text{Spec}(R)$ into a ringed space.

Proof idea. For any open set U , define $\mathcal{O}(U)$ to be the functions f on U with the property that for any $[\mathfrak{p}] \in U$, $f([\mathfrak{p}]) \in R_{\mathfrak{p}}$ and there exists an open set $V \subseteq U$ containing $[\mathfrak{p}]$ and elements $r, s \in R$ where $s \notin \mathfrak{q}$ for any $[\mathfrak{q}] \in V$ and $f([\mathfrak{q}]) = \frac{r}{s} \in R_{\mathfrak{q}}$ for all $[\mathfrak{q}] \in V$. Need to check that this is indeed a sheaf of rings. \square

Lemma 129. The sheaf \mathcal{O} defined as above makes $\text{Spec}(R)$ into a locally ringed space, namely, the stalk at $[\mathfrak{p}]$ is a local ring for any prime ideal \mathfrak{p} .

Proof. We claim that $\mathcal{O}_{[\mathfrak{p}]} \cong R_{\mathfrak{p}}$ which is a local ring. \square

Proposition 130. Let $f \in R$, then $\mathcal{O}(D(f)) = R[f^{-1}]$.

Corollary 131. $\mathcal{O}(\text{Spec}(R)) = R$.

Definition 132 (Morphism of sheaves). A morphism of sheaves $\phi : \mathcal{O} \rightarrow \mathcal{P}$ is a natural transformation between the two functors.

Definition 133 (Morphism of ringed spaces). Let (X, \mathcal{O}_X) and (Y, \mathcal{O}_Y) be ringed spaces and $f : X \rightarrow Y$ is a continuous map. We can define a new sheaf on Y $f_*\mathcal{O}_X = U \mapsto \mathcal{O}_X(f^{-1}(U))$. A morphism of ringed spaces is f along with a morphism of sheafs $f^\# : \mathcal{O}_Y \rightsquigarrow f_*\mathcal{O}_X$. If X and Y are locally ringed spaces, $(f, f^\#)$ is a morphism of locally ringed spaces if the induced ring homomorphisms $f^\# : \mathcal{O}_{Y, f(x)} \rightarrow \mathcal{O}_{X, x}$ is a local homomorphism. Namely, the preimage of the maximal ideal is maximal.

Definition 134 (Affine schemes). An affine scheme is a locally ringed space isomorphic to $(\text{Spec}(R), \mathcal{O})$ for some ring R . The category of affine schemes is a full subcategory of the category of locally ringed spaces.

Theorem 135. *The category of commutative rings is anti-equivalent to the category of affine schemes.*