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Important Results - MATH 350

Ralph Sarkis 260729917

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1 Introduction to graphs

Lemma 1.1 (Handshaking lemma). *For every graph $G = (V, E)$, the sum of the degrees of all the vertices is even.*

Corollary 1.2. *The number of vertices with odd degrees is even.*

Proposition 1.3. *For any two vertices, the existence of walk between them guarantees the existence of a trail which guarantees the existence of a path which guarantees the existence of a walk.*

Corollary 1.4. *For any graph G, the walk, trail and path relations are equivalence relations on $V(G)$.*

Lemma 1.5. *Let G be a graph, $e \in E(G)$ is a cut edge if and only if there is no cycle in G containing e.*

1.1 From Assignments

Proposition 1.6. *Let $G = (V, E)$ be a simple graph with $|V| \geq 2$, then $\exists v, w \in V, \deg(v) = \deg(w)$.*

Proposition 1.7. *Let G be a disconnected graph, the complement of G, $\overline{G}$ is connected.*

Proposition 1.8. *Let G be a graph with minimum degree k, then G contains a cycle of length k.*

2 Trees

Lemma 2.1. *Every tree with at least two vertices has at least two leafs.*

Corollary 2.2. *Let G be a graph. For any leafs $v \in V(G)$, G is tree if and only if $G - v$ is a tree.*

Proposition 2.3. *A graph G being a tree is equivalent to each of the following statements:*

1. *G is connected and contains no cycle*
2. *$\forall e \in E(G)$, e is a cut-edge*
3. *G is connected and every trail in G is a path*
4. *Between any two vertices there is a unique path.*
5. *Maximal graph with respect to adding edges that has no cycle*
6. *G is connected and $|V(G)| = |E(G)| + 1$*
7. *G has no cycle and $|V(G)| = |E(G)| + 1$*

Lemma 2.4. *For every rooted trees, there exists a unique out-rooted orientation.*

Theorem 2.5 (Cayley's formula)**.** *We denote t_n to be the number of labeled trees on $\{1, \ldots, n\}$.*

$$t_n = n^{n-2}$$

2.1 From Assignments

Proposition 2.6. *If a tree T contains a vertex of degree k, then T has at least k leaves.*

3 Spanning Trees

Proposition 3.1. *If G is connected, then G has a spanning tree.*

Proposition 3.2. *Let T be the spanning tree of a graph G and $e \in E(G) \setminus E(T)$, take any edge f in the fundamental cycle with respect to T and e. Then, $T_p = (T + e) - f$ is a spanning tree.*

Algorithm 3.3 (Kruskal)**.** Kruskal gives a greedy algorithm to find the shortest path spanning tree of a graph. Let $G = (V, E)$ be a graph and w be a weight function on it.

KRUSKAL$(G = (V, E), w)$

Initialize $T = (V, \emptyset)$
for each $e = \{u, v\}$ in E sorted by increasing weight **do**
    if $u \not\sim v$ **then**
        Add e to T.
return T

Algorithm 3.4 (Dijkstra)**.** Dijkstra gives a greedy algorithm to find the path of minimum between two vertices. Let $G = (V, E)$ be a graph, w a weight function on it and s and t be source and target vertices.

DIJKSTRA$(G = (V, E), w, s, t)$

Initialize $T = (V, \emptyset)$
Initialize $\text{dist}[u] = \infty$ for all $u \in V \setminus \{s\}$. Set $\text{dist}[s] = 0$
Initialize $H = \{s\}$ a min-heap of vertices sorted by dist
while $H \neq \emptyset$ **do**
    Let $u = H.remove\_min()$
    Add the edge of smallest weight connecting u to T.
    for each neighbor v of u **do**
        Set $\text{dist}[v] = \min\{\text{dist}[v], \text{dist}[u] + w(\{u, v\})\}$
return T

4 Euler Tours

Theorem 4.1. *A multigraph G contains a closed Eulerian tour if and only if G is connected and there is no vertices of odd degree.*

Corollary 4.2. *A multigraph G contains an Eulerian tour if and only if it is connected and contains at most two vertices of odd degree.*

Theorem 4.3 (Ore's theorem)**.** *Let $G = (V, E)$ be a graph with $n = |V| \geq 3$. Suppose that for every pair $u, w \in V$ such that $\{u, w\} \notin E$, $\deg(u) + \deg(w) \geq n$, then G contains a Hamiltonian cycle.*

Corollary 4.4. *Let $G = (V, E)$ be a graph, then $\min_{v \in V} \deg(v) \geq \frac{n}{2}$ implies that G contains a Hamiltonian cycle.*

5 Bipartite Graphs

Theorem 5.1. *A graph G is bipartite if and only if it has no odd cycle.*

Theorem 5.2. *Let $G = (V, E)$ be a graph, then the following are equivalent :*

1. *G is bipartite*
2. *G does not contain a closed walk of odd length*
3. *G does not contain an odd cycle*

Proposition 5.3. *Let G be a simple graph. G is bipartite if and only if it contains no induced cycle of odd length.*

6 Matching in graphs

Proposition 6.1. *For any $k \geq 1$, a (2^k)-regular graph contains a 2-factor.*

Lemma 6.2 (Berge)**.** *Let $G = (V, E)$ be a graph and M be a matching in G. M is maximum matching if and only if there is no M-augmenting paths.*

Theorem 6.3 (Konig)**.** *Let G be a bipartite graph, then $\tau(G) = \nu(G)$.*

Theorem 6.4 (Hall)**.** *Let G be a bipartite graph with bipartition A and B, then there exists an A-covering matching in G if and only if $\forall S \subseteq A, |N(S)| \geq |S|$.*

Theorem 6.5. *Every $(2k)$-regular graph has a 2-factor.*

Corollary 6.6. *Every $(2k)$-regular graph has k disjoint 2-factors.*

Proposition 6.7. *Let $G = (V, E)$ be any graph, $\alpha(G) + \tau(G) = |V|$.*

Proposition 6.8 (Gallai)**.** *Let $G = (V, E)$ be any graph, $\rho(G) + \nu(G) = |V|$.*

Corollary 6.9. *If G is bipartite, $\alpha(G) = \rho(G)$.*

Theorem 6.10 (Tutte)**.** *Let $G = (V, E)$ be any graph, then G has a perfect matching if and only if for any subset of vertices X, $\mathrm{Odd}(G - X) \leq |X|$.*

Theorem 6.11 (Petersen)**.** *All 3-regular graphs containing no cut-edges have perfect matchings.*

Corollary 6.12. *A 3-regular graph G has a perfect matching if and only if it has a 2-factor.*

Lemma 6.13. *Let $G = (V, E)$ be a graph with $|V|$ even. Then for any $X \subsetneq V$, $\mathrm{Odd}(G - X) \equiv |X| \mod 2$.*

Corollary 6.14. *Let $G = (V, E)$ be a bipartite graph with parts A and B. Suppose that $\forall S \subseteq A, |N_G(S)| \geq |S|$, then G has an A-covering matching.*

7 Ramsey Theory

Lemma 7.1.

$$R(k, \ell) \leq R(k-1, \ell) + R(k, \ell - 1)$$

Lemma 7.2.

$$R(k, \ell) \leq \binom{k + \ell - 2}{k - 1}$$

Corollary 7.3. $R(k) = R(k, k) < 4^k$

Theorem 7.4 (Ramsey)**.** *For any $k \in \mathbb{N}, R(k) < 4^k$, implying $R(k)$ is finite.*

Theorem 7.5. *For any ℓ and k, $R_\ell(k)$ is finite.*

Theorem 7.6 (Schur)**.** *For any $\ell \in \mathbb{N}$, $\mathbb{N}^+$ is not ℓ-colorable such that $x + y = z$ has no monochromatic solution.*

8 Connectivity of Graphs

Theorem 8.1 (Menger). *Let G be a simple and not complete graph, then*

$$\kappa(G) = \min_{C \text{ vertex cut}} |C|$$

Theorem 8.2 (Ford-Fulkerson). *Let G be a multigraph, then*

$$\kappa'(G) = \min_{F \text{ edge cut}} |F|$$

Lemma 8.3. *Let $G = (V, E)$ be a simple graph. For any $u \neq w \in V$ such that $\{u, w\} \notin E$, we have $c_G(u, w) = P_G(u, w)$.*

Lemma 8.4. *Let $G = (V, E)$ be a multigraph. For any $u \neq w \in V$, we have $c'_G(u, w) = P'_G(u, w)$.*

9 Networks

Lemma 9.1. *Let $D = (V, E)$ be a digraph and $s \neq t \in V$, then either there exists a directed path from s to t or there exists a subset $X \subseteq V$ with $\{s\} \subseteq X \subseteq V \setminus \{t\}$ such that $\partial^+(X) = \emptyset$.*

Lemma 9.2. *Let $D = (V, E)$ be a digraph, $s \neq t$ be vertices and ϕ be an s, t-flow of value k, then $\forall \{s\} \subseteq X \subseteq V \setminus \{t\}$, we have*

$$\sum_{e \in \partial^+(X)} \phi(e) - \sum_{e \in \partial^-(X)} \phi(e) = k$$

Lemma 9.3. *Let $D = (V, E)$ be a digraph, $s \neq t$ be vertices and ϕ be an integral s, t-flow of value k, then there exists a collection of paths $\{P_1, \dots, P_k\}$ all going from s to t such that every edge $e \in E$ belongs to at most $\phi(e)$ paths.*

Lemma 9.4. *Let (V, E, s, t, c) be a network, ϕ be an integral c-admissible s, t-flow and P be an augmenting path for ϕ. Then there exists a c-admissible s, t-flow ψ with* $\mathrm{val}(\psi) \geq \mathrm{val}(\phi) + 1$.

Theorem 9.5 (Ford-Fulkerson). *Let (V, E, s, t, c) be a network and Φ be the set of all c-admissible s, t-flows, then, we have the following:*

$$\max_{\phi \in \Phi} \mathrm{val}(\phi) = \min_{\{v\} \subseteq X \subseteq V \setminus \{t\}} \mathrm{cap}(X)$$

10 Proper Vertex Coloring

Proposition 10.1. *Let G be a simple graph, recall that $\alpha(G)$ is the size of the largest independent set. We have $\chi(G) \geq \frac{|V(G)|}{\alpha(G)}$.*

Theorem 10.2. *For any $k \in \mathbb{N}$, there exists a graph G_k simple graph with no triangles and $\chi(G_k) > k$.*

Theorem 10.3. *Let $G = (V, E)$ be a graph without triangles with $n = |V|$, then $\chi(G) \leq \sqrt{2n}$.*

Theorem 10.4 (Brooks)**.** *Let G be a connected loopless multigraph that is not complete nor an odd cycle, then $\chi(G) \leq \Delta(G)$.*

Proposition 10.5. *If G is d-degenerate, then $\chi(G) \leq d + 1$.*

Theorem 10.6 (Vizing)**.** *If G is a simple graph, then $\chi'(G) \leq \Delta(G) + 1$. If G is not simple, then denote $\mu(G)$ to be the maximum multiplicity of an edge in G, we have $\chi'(G) \leq \Delta(G) + \mu(G)$.*

Theorem 10.7 (Konig's line coloring)**.** *If $G = (V, E)$ is a bipartite graph, then $\chi'(G) = \Delta(G)$.*

Theorem 10.8 (Shannon)**.** *If G is a loopless multigraph, $\chi'(G) \leq 3\lceil \frac{\Delta(G)}{2} \rceil$.*

10.1 From Assignments

Proposition 10.9. *Let G be a simple graph and $\overline{G}$ be its complement, then $\chi(G)\chi(\overline{G}) \geq |V|$.*

Proposition 10.10. *Let G be a simple graph such that for any two odd cycles C_1 and C_2, $V(C_1) \cap V(C_2) \neq \emptyset$, then $\chi(G) \leq 5$.*

Proposition 10.11. *Let G be a 3-regular simple graph with a Hamiltonian cycle, then $\chi'(G) = 3$.*

Proposition 10.12.

$$\chi'(K_n) = \begin{cases} n & \text{for odd } n \\ n-1 & \text{for even } n \end{cases}$$

11 Structural Graph Theory

Theorem 11.1 (Jordan's curve theorem)**.** *Any continuous non self-intersecting loop in the plane divides the plane in exactly two regions.*

Lemma 11.2. *If the top and bottom regions of an edge are the same, then it must be a cut edge.*

Theorem 11.3 (Euler's formula)**.** *Let D be a drawing of $G = (V, E)$ a connected planar graph, then $|V| + \mathrm{Reg}(D) - |E| = 2$, where $\mathrm{Reg}(D)$ denotes the number of regions in D.*

Corollary 11.4. *If e is a cut edge of G, then e is surrounded by only one region in any drawing of G.*

Proposition 11.5. *Let G be a planar graph and D be an arbitrary plane drawing, then*

$$\sum_{R \text{ region in } D} \ell(R) = 2|E(G)|$$

Corollary 11.6. *Let G be a planar graph and D_1 and D_2 be two of its plane drawings, then*

$$\sum_{R \text{ region in } D_1} \ell(R) = \sum_{R \text{ region in } D_2} \ell(R)$$

Theorem 11.7. *Let $G = (V, E)$ be a simple planar graph with $n = |V| \geq 3$, $m = |E|$ and $f = \mathrm{Reg}(G)$, then $m \leq 3n - 6$.*

Corollary 11.8. *K_5 is not planar.*

Lemma 11.9. *Let G be a connected planar graph with $n \geq 3$ vertices. For any drawing D and any region R in D, $\ell(R) \geq 3$.*

Lemma 11.10. *G is planar if and only if $G \,\mathrm{sbd}\, e$ is planar.*

Proposition 11.11. *$K_{3,3}$ is not planar.*

Theorem 11.12 (Kuratowski)**.** *G is planar if and only if G contains no subdivision of K_5 or $K_{3,3}$.*

Proposition 11.13. *Let G be a planar graph, $e \in E$ an edge and G' be the contraction of e, then G' is planar.*

Theorem 11.14 (Kuratowski-Wagner)**.** *G is planar if and only if G does not contain K_5 nor $K_{3,3}$ as a minor.*

12 Coloring of Planar Graphs

Theorem 12.1. *Every planar graph can be drawn using straight lines only.*

Theorem 12.2 (Four color theorem)**.** *Let G be a planar graph, then $\chi(G) \leq 4$.*

Theorem 12.3 (Six color theorem)**.** *Let G be a planar graph, then $\chi(G) \leq 6$.*

Theorem 12.4 (Five color theorem)**.** *Let G be a planar graph, then $\chi(G) \leq 5$.*

Theorem 12.5. *Let G be a K_5 minor-free graph, then $\chi(G) \leq 4$.*

Theorem 12.6. *Let G be a K_4 minor-free graph, then $\chi(G) \leq 4$.*

Theorem 12.7. *Let G be a K_3 minor-free graph, then $\chi(G) \leq 2$.*

Theorem 12.8. *Let G be a K_4 minor-free graph, then $m \leq 2n - 3$, where $m = |E|$ and $n = |V|$.*

12.1 From Assignments

Proposition 12.9. *Let G be a simple triangle-free graph, then* $\chi(G) \leq 4$.

Proposition 12.10. *Let G be an outerplanar graph, then* $\chi(G) \leq 3$.

Proposition 12.11. *A graph G is outerplanar if and only if it does not contain* K_4 *nor* $K_{3,3}$ *as a minor.*

Proposition 12.12. *Let H be a simple graph with maximum degree at most 3. Show that every simple graph contains a subdivision of H if and only if it contains H as a minor.*

Proposition 12.13. *Let G be a simple graph that contains* K_5 *as a minor, then G contains a subdivision of of* K_5 *or a subdivision of* $K_{3,3}$.