Quantitative Algebraic Theories for Monad Liftings

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Abstract

Quantitative equational logic introduced by Mardare, Panangaden and Plotkin has enabled the algebraic axiomatisation of many metrics. This is achieved by finding a quantitative algebraic theory that presents a monad on a category of metric spaces. We show how to construct such a theory for monads that lift monads on **Set** with a known algebraic presentation.

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1 Introduction

Moggi's seminal papers [20, 21] started a long-lasting tradition in the field of denotational semantics of modelling computational effects with monads. Examples include nondeterminism, probabilistic nondeterminism, input/output, termination, exceptions, and more.

Monads are largely indispensable in category theory, and they had been extensively studied by practitioners of the field before that. In particular, in his thesis [12], Lawvere identified a tight link between monads and universal algebra. Every algebraic theory presents a monad on the category of sets (**Set**), and every finitary (a categorical finiteness property) monad on **Set** is presented by an algebraic theory. In an excellent series of papers [22, 23, 24, 25], Plotkin and Power advocated for exploiting this link and studying computational effects through algebraic theories. We refer you to the survey [11] and tutorial [6] for more.

The framework of quantitative algebraic reasoning was introduced in [14] as a natural extension of previously cited works to reason about program distances instead of program equivalences. It quickly spanned a lot of theoretical [15, 4, 13, 5, 1, 18, 2, 10] and practical investigations [3, 19, 17], and our paper follows their lead.

Given a signature Σ (a set of operation symbols with finite arities), a quantitative Σ–algebra is a metric space (A, d_A) equipped with interpretations of the symbols in Σ as functions $A^n \to A$ with possible extra conditions. We will notably impose no conditions on the operations. This extends to an interpretation of all terms formed with finitely many applications of operation symbols in Σ.

A quantitative algebraic theory over a signature Σ is a class $\widehat{E}$ of equations and so-called quantitative equations between terms formed over variables. As in the classical case, an equation $s = t$ means that s and t are interpreted as the same thing. A *quantitative* equation $s =_\varepsilon t$ is parameterised by $\varepsilon \in [0, 1]$, and it means that the distance between the interpretation of s and t is at most ε. Given a theory $(\Sigma, \widehat{E})$, the free quantitative $(\Sigma, \widehat{E})$–algebra on a metric space (X, d) is constructed by taking all the terms over X, defining the distance between s and t to be the smallest distance that can be derived using $\widehat{E}$, and quotienting by the equations that are deductible from $\widehat{E}$. This induces a monad $\widehat{T}_{\Sigma,\widehat{E}}$ on **Met**, the category of metric spaces and nonexpansive maps. We say a monad on **Met** is presented by $(\Sigma, \widehat{E})$ if it is isomorphic to $\widehat{T}_{\Sigma,\widehat{E}}$. Not all monads on **Met** have a presentation, but a characterization of those that do is well under way [1, 2].

Most concrete results in the literature give presentations for monads on **Met** that take great inspiration from presentations of monads on **Set**. For instance, two monads considered in the original paper [14] are the finite powerset (with the Hausdorff distance) monad and the finite distributions (with the Kantorovich distance) monad on **Met**. They are monad liftings (see Definition 27) of existing monads on **Set** that are presented by the algebraic theories of semilattices and convex algebras respectively. The latter are key ingredients for their presentation results.

Contributions. In this paper, we prove (Theorem 32) that if a monad $\widehat{M}$ on **Met** is a monad lifting of a monad M on **Set** presented by an algebraic theory (Σ, E), then $\widehat{M}$ is presented by a quantitative algebraic theory. The proof explicitly constructs that theory using Σ and E, so it can be seen as a more automatic way to apply the quantitative algebraic reasoning framework. Another consequence of this result is Corollary 33 that gives a correspondence between monad liftings of M and theory liftings (Definition 28) of E. Finally, we show in Section 5 how to simplify the proofs of existing presentation results using Theorem 32, and we give two new presentation results.

As mentioned above, our treatment of quantitative algebras and theories is different from most of the literature because operations are not assumed to be nonexpansive with respect to the product metric. This idea borrowed and altered from [18] is necessary for Theorem 32 to hold. Indeed, we define a monad lifting of the finite powerset monad to **Met** (Proposition 34) that cannot be presented by a theory in the sense of [14], in short because it is not enriched. As in [18], our results also apply more generally to variants of **Met**, like the category of pseudometric spaces, quasimetric spaces and more (see Definition 11).

2 Background

We recall some definitions and results following the background section of [18] but with a slightly different presentation tailored for our usage. Facts easily derivable from known results in the literature are systematically marked as "Proposition" throughout the paper.

2.1 Monads

▶ **Definition 1.** *A* monad *on a category* $\mathbf{C}$ *is a triple* (M, η, μ) *comprising a functor* $M \colon \mathbf{C} \to \mathbf{C}$ *together with two natural transformations: a* unit $\eta \colon \mathrm{id}_{\mathbf{C}} \Rightarrow M$*, where* $\mathrm{id}_{\mathbf{C}}$ *is the identity functor on* $\mathbf{C}$*, and a* multiplication $\mu \colon M^2 \Rightarrow M$*, satisfying* $\mu \circ \eta M = \mu \circ M\eta = \mathrm{id}_M$ *and* $\mu \circ M\mu = \mu \circ \mu M$*.*

We often refer to a monad by simply specifying the functor. A monad M has an associated category of M–algebras.

▶ **Definition 2.** *Let* (M, η, μ) *be a monad on* $\mathbf{C}$*. An* algebra *for* M *(or* M–algebra*) is a pair* (A, α) *where* $A \in \mathbf{C}$ *is an object and* $\alpha : M(A) \to A$ *is a morphism such that (1)* $\alpha \circ \eta_A = \mathrm{id}_A$ *and (2)* $\alpha \circ M\alpha = \alpha \circ \mu_A$ *hold. An* M–algebra morphism *between two* M*–algebras* (A, α) *and* (A', α') *is a morphism* $f : A \to A'$ *in* $\mathbf{C}$ *such that* $f \circ \alpha = \alpha' \circ M(f)$*. The category of* M*–algebras and their morphisms, denoted by* $\mathbf{EM}(M)$*, is called the Eilenberg–Moore category for* M*. There is a forgetful functor* $U : \mathbf{EM}(M) \to \mathbf{C}$ *that forgets the algebra structures.*

▶ **Definition 3.** *Let* (M, η, μ) *and* (M', η', μ') *be two monads on* $\mathbf{C}$*. A* monad morphism *from* M *to* M' *is a natural transformation* $\lambda : M \Rightarrow M'$ *such that (1)* $\lambda \circ \eta^M = \eta^{M'}$ *and (2)* $\lambda \circ \mu^M = \mu^{M'} \circ \lambda M' \circ M\lambda$*. It is a* monad isomorphism *whenever each component* $\lambda_X : MX \to M'X$ *is an isomorphism in* $\mathbf{C}$*.*

▶ **Proposition 4.** *Let* (M, η, μ) *and* (M', η', μ') *be two monads on* $\mathbf{C}$*. There is a monad isomorphism* $M \cong M'$ *if and only if there is an isomorphism of categories* $\mathbf{EM}(M) \cong \mathbf{EM}(M')$ *that commutes with the forgetful functors to* $\mathbf{C}$*.*

2.2 Universal Algebra

We recall basic definitions and results from universal algebra, [7] is a standard reference.

▶ **Definition 5** (Signature)**.** *A* signature *is a set* Σ *containing operations symbols each with an arity* $n \in \mathbb{N}$*. We write* $\mathsf{op} : n \in \Sigma$ *for a symbol* op *with arity* n *in* Σ*. With some abuse of notation, we also denote by* Σ *the functor* $\Sigma : \mathbf{Set} \to \mathbf{Set}$ *with the following action:*

$$\Sigma(A) := \coprod_{\mathsf{op}:n\in\Sigma} A^n \quad \Sigma(f) := \coprod_{\mathsf{op}:n\in\Sigma} f^n.$$

▶ **Definition 6** (Σ–algebra)**.** *A* Σ–algebra *is an algebra for the functor* Σ*. Equivalently, it is a set* A *equipped with a set* $[\![\Sigma]\!]_A$ *of interpretations of the operation symbols, i.e., for every* $\mathsf{op} : n \in \Sigma$ *there is a function* $[\![\mathsf{op}]\!]_A : A^n \to A$ *in* $[\![\Sigma]\!]_A$*. We call* A *the* carrier set*. A* homomorphism *between two* Σ*–algebras with carrier sets* A *and* B *is a function* $f : A \to B$ *preserving the interpretations of operations, i.e., satisfying* $\forall \mathsf{op} : n \in \Sigma, \forall a_1, \dots, a_n$*,*

$$f([\![\mathsf{op}]\!]_A(a_1, \dots, a_n)) = [\![\mathsf{op}]\!]_B(f(a_1), \dots, f(a_n)).$$

The category of Σ*–algebras and their homomorphisms is denoted by* $\mathbf{Alg}(\Sigma)$*.*

▶ **Definition 7** (Term)**.** *Let* Σ *be a signature and* A *be a set. We denote with* $T_\Sigma A$ *the set of* terms *built from* A *using the operations in* Σ*, i.e., the set inductively defined as follows:* $a \in T_\Sigma A$ *for any* $a \in A$*, and* $\mathsf{op}(t_1, \dots, t_n) \in T_\Sigma A$ *for any* $\mathsf{op} : n \in \Sigma$ *and* $t_1, \dots t_n \in T_\Sigma A$*. We often identify elements* $a \in A$ *with the corresponding terms* $a \in T_\Sigma A$*. In any* Σ*–algebra* $(A, [\![\Sigma]\!]_A)$*, we can extend the interpretations of operations to all terms in* $T_\Sigma A$ *inductively:*

$$[\![a]\!]_A = a \text{ and } [\![\mathsf{op}(t_1, \dots, t_n)]\!]_A = [\![\mathsf{op}]\!]_A([\![t_1]\!]_A, \dots, [\![t_n]\!]_A).$$

The assignment $A \mapsto T_\Sigma A$ *can be turned into a functor* $T_\Sigma : \mathbf{Set} \to \mathbf{Set}$ *by inductively defining, for any function* $f : A \to B$*, the function* $T_\Sigma f : T_\Sigma A \to T_\Sigma B$ *as follows: for any* $a \in A$*,* $T_\Sigma f(a) = f(a)$*, and* $\forall \mathsf{op} : n \in \Sigma, \forall t_1, \dots t_n \in T_\Sigma A$*,*

$$T_\Sigma f(\mathsf{op}(t_1, \dots, t_n)) = \mathsf{op}(T_\Sigma f(t_1), \dots, T_\Sigma f(t_n)).$$

This allows to extend the interpretation $[\![-]\!]_A$ *to all terms in* $T_\Sigma X$ *provided we have an assignment of variables* $\iota : X \to A$ *by precomposing with* $T_\Sigma \iota$*. We denote this interpretation* $[\![-]\!]^\iota_A = [\![-]\!]_A \circ T_\Sigma \iota$*.*

▶ **Definition 8** (Equation)**.** *An* equation *over* Σ *is a triple comprising a set* X *of variables, also called* context*, and a pair of terms* $s, t \in T_\Sigma X$ *that we denote by* $\forall X.s = t$ *following [7]. The symbol* $\forall$ *does not indicate a quantification over* X*, but over assignments of variables as explained below. We say that an equation* $\forall X.s = t$ *is* satisfied *in a* Σ*–algebra* $\mathbb{A} = (A, [\![\Sigma]\!]_A)$*, and we write* $\mathbb{A} \vDash \forall X.s = t$*, if for all assignments of variables* $\iota : X \to A$*,* $[\![s]\!]^\iota_A = [\![t]\!]^\iota_A$*.*

Given a class E *of equations over* Σ*, we write* $\mathbb{A} \vDash E$ *if* $\mathbb{A}$ *satisfies all equations in* E*, and we denote by* $\mathbf{Alg}(\Sigma, E)$ *the full subcategory of* $\mathbf{Alg}(\Sigma)$ *of all* Σ*–algebras that satisfy all equations in* E*.*

▶ **Definition 9** (Algebraic theory). *Given a class E of equations over Σ, $\mathbf{Th}(\mathbf{Alg}(\Sigma, E))$ is the class of equations that are satisfied in all algebras in $\mathbf{Alg}(\Sigma, E)$. Of course, $\mathbf{Th}(\mathbf{Alg}(\Sigma, E))$ contains all equations in E, but also many more equations like $\forall x.x = x$ which is satisfied by any algebra in $\mathbf{Alg}(\Sigma)$. An* algebraic theory *is a class E of equations over a signature Σ such that $E = \mathbf{Th}(\mathbf{Alg}(\Sigma, E))$. For any set of equations E, $\mathbf{Th}(\mathbf{Alg}(\Sigma, E))$ is an algebraic theory, and we call equations in E the* generators *of this theory.*

▶ **Proposition 10.** *For any algebraic theory (Σ, E), the forgetful functor $U : \mathbf{Alg}(\Sigma, E) \to \mathbf{Set}$ that forgets about the algebra structure is strictly monadic.*

Proof sketch. We give the detailed constructions of the left adjoint via free algebras because they will be used in the rest of the paper. Given a set X, the carrier of the free (Σ, E)–algebra on X is the set of terms in $T_\Sigma X$ quotiented by the equivalence relation

$$s \equiv_E t \Leftrightarrow \forall X.s = t \in E.$$

We denote by $[s]_E$ the equivalence class of $s \in T_\Sigma X$ in $T_{\Sigma,E}X := T_\Sigma X/{\equiv_E}$. The interpretation of $\mathsf{op} : n \in \Sigma$ is defined syntactically (a bit of work is needed to show this is well-defined):

$$[\![\mathsf{op}]\!]([t_1]_E, \dots, [t_n]_E) = [\mathsf{op}(t_1, \dots, t_n)]_E.$$

The universal morphism from X to U is $\eta^{\Sigma,E}_X : X \to T_\Sigma X/{\equiv_E}$ sending x to $[x]_E$. After showing U uniquely creates coequalizers of U–split pairs, we obtain a monad $T_{\Sigma,E}$ with unit $\eta^{\Sigma,E}$ and multiplication $\mu^{\Sigma,E}$ such that $\mathbf{EM}(T_{\Sigma,E}) \cong \mathbf{Alg}(\Sigma, E)$. The explicit definitions of $T_{\Sigma,E}$ applied to $f : A \to B$ and the multiplication are respectively

$$T_{\Sigma,E}f : T_{\Sigma,E}A \to T_{\Sigma,E}B = [t]_E \mapsto [T_\Sigma f(t)]_E, \text{ and}$$
$$\mu^{\Sigma,E}_X : T_{\Sigma,E}T_{\Sigma,E}X \to T_{\Sigma,E}X = [t([t_1]_E, \dots, [t_n]_E)]_E \mapsto [t(t_1, \dots, t_n)]_E.$$

Let us also explicit the isomorphism between the categories of algebras.

Given a $T_{\Sigma,E}$–algebra $\alpha : T_{\Sigma,E}A \to A$, we define $[\![\Sigma]\!]_A$ by letting $[\![\mathsf{op}]\!]_A(a_1, \dots, a_n) = \alpha([\mathsf{op}(a_1, \dots, a_n)]_E)$ for each $\mathsf{op} : n \in \Sigma$. Given a Σ–algebra $\mathbb{A} = (A, [\![\Sigma]\!]_A)$ that satisfies E, the interpretations of terms $[\![-]\!]_A : T_\Sigma A \to A$ identifies terms equivalent under $\equiv_E$ (by definition of satisfaction). Thus, $[\![-]\!]_A$ factorises through $T_{\Sigma,E}A$, and one can show the resulting function $\alpha_{\mathbb{A}} : T_{\Sigma,E}A \to A$ is a $T_{\Sigma,E}$–algebra. ◀

2.3 Generalised Metric Spaces

The literature on quantitative algebraic reasoning is mostly focused on the category $\mathbf{Met}$ of metric spaces (with possibly infinite distances or distances bounded by 1) and nonexpansive maps. In continuity with [18, Section 2.3], we allow for many variants of $\mathbf{Met}$ that we call $\mathbf{GMet}$ (more details are in *loc. cit.*).

▶ **Definition 11 (GMet).** *A* generalised metric space *is a set X equipped with a distance function $d : X \times X \to [0, 1]$ that satisfies some axioms, e.g. symmetry, triangle inequality, etc. For a fixed set of axioms, we denote by $\mathbf{GMet}$ the category of generalised metric spaces that satisfy these axioms with morphisms being nonexpansive maps.*

The category $\mathbf{Met}$ is an instance of $\mathbf{GMet}$ where the axioms are

$$\forall a, b \in A, \quad d(a, b) = d(b, a) \qquad \text{symmetry} \tag{1}$$
$$\forall a \in A, \quad d(a, a) = 0 \qquad \text{reflexivity or indiscernibility of identicals} \tag{2}$$
$$\forall a, b \in A, \quad d(a, b) = 0 \implies a = b \qquad \text{indentity of indiscernibles} \tag{3}$$
$$\forall a, b, c \in A, \quad d(a, c) \le d(a, b) + d(b, c). \qquad \text{triangle inequality} \tag{4}$$

In the sequel, the instantiation of $\mathbf{GMet}$ will not play an important role. In fact, almost all examples will be in $\mathbf{Met}$. Hence, for a better reading experience, throughout the sequel, we fix an arbitrary instance of $\mathbf{GMet}$ and refer to its objects as metric spaces (omitting the word "generalised").

For any set X, the discrete generalised metric on X is a distance $d_\perp : X \times X \to [0,1]$ satisfying the axioms of $\mathbf{GMet}$ such that for any (Y, Δ) and any function $f : X \to Y$, $f : (X, d_\perp) \to (Y, \Delta)$ is nonexpansive. This can also be stated as a universal property, so $(X, d_\perp)$ is unique with this property, and the assignment $X \mapsto (X, d_\perp)$ assembles into a functor $F_\perp : \mathbf{Set} \to \mathbf{GMet}$ left adjoint to the forgetful functor $U : \mathbf{GMet} \to \mathbf{Set}$.

3 Quantitative Algebras and (Generalised) Metric Monads

This section presents our framework for quantitative algebraic reasoning which is slightly different from the original [14]. It borrows from [18] the generalisation to $\mathbf{GMet}$ and not non-expansive operations, and from [9] the handling of context for (quantitative) equations (called Σ–relations in *loc. cit.*). We define quantitative algebras and their equations, quantitative theories and the monads they induce, and we define algebraic presentations. In contrast to the previous references, we omit the syntactical deductive system, but we prove many small results that essentially amount to soundness with respect to that hypothetical deductive system, as they are useful in proofs.

Quantitative Algebras

Given a signature Σ, we abusively denote by Σ the functor $\Sigma : \mathbf{GMet} \to \mathbf{GMet}$ defined by the composite $\mathbf{GMet} \xrightarrow{U} \mathbf{Set} \xrightarrow{\Sigma} \mathbf{Set} \xrightarrow{F_\perp} \mathbf{GMet}$, it has the following action:

$$\Sigma(A, d) := \left(\coprod_{\mathsf{op}:n \in \Sigma} A^n, d_\perp \right) \qquad \Sigma(f) := \coprod_{\mathsf{op}:n \in \Sigma} f^n.$$

▶ **Definition 12** (Quantitative algebra). *A* $\mathbf{GMet}$ Σ–algebra *is an algebra for the functor* $\Sigma : \mathbf{GMet} \to \mathbf{GMet}$*. Equivalently, it is a metric space* (A, d) *equipped with a set* $[\![\Sigma]\!]_A$ *of interpretations of the operation symbols, i.e., for every* $\mathsf{op} : n \in \Sigma$ *there is a function* $[\![\mathsf{op}]\!]_A : A^n \to A$ *in* $[\![\Sigma]\!]_A$*. We call* (A, d) *the* carrier space*. A* homomorphism *between two* Σ*–algebras with carrier spaces* (A, d_A) *and* (B, d_B) *is a nonexpansive function* $f : (A, d_A) \to (B, d_B)$ *preserving the interpretations of operations, i.e., satisfying* $\forall \mathsf{op} : n \in \Sigma, \forall a_1, \ldots, a_n$*,*

$$f([\![\mathsf{op}]\!]_A(a_1, \ldots, a_n)) = [\![\mathsf{op}]\!]_B(f(a_1), \ldots, f(a_n)).$$

The category of $\mathbf{GMet}$ Σ*–algebras and their homomorphisms is denoted by* $\mathbf{QAlg}(\Sigma)$*.*

▶ Remark 13. When the category $\mathbf{GMet}$ is irrelevant (or when it is fixed as in this paper), we use the term quantitative algebra as in [14] and [18]. The difference between Definition 12 and analogous definitions in those papers is that we impose no condition on the operations. This can be seen as a special case of [18] since $\Sigma : \mathbf{GMet} \to \mathbf{GMet}$ can be seen as the lifted signature [18, Definition 3.6] of $\Sigma : \mathbf{Set} \to \mathbf{Set}$ where all operations are lifted with the discrete metric. We do not loose generality because the condition on operations imposed by lifted signatures can be imposed by sets of (quantitative) equations (see Definition 14).

Any quantitative Σ–algebra $(A, d_A, [\![\Sigma]\!]_A)$ has an underlying Σ–algebra $(A, [\![\Sigma]\!]_A)$ in $\mathbf{Alg}(\Sigma)$ and a carrier space (A, d_A) in $\mathbf{GMet}$. We get two more forgetful functors and the pullback square in (5). In particular, we can still talk about terms and their interpretations inside a quantitative algebra, and we can define $\mathbf{GMet}$ equations.

$$\begin{tikzcd} \mathbf{QAlg}(\Sigma) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{GMet} \arrow[d] \\ \mathbf{Alg}(\Sigma) \arrow[r] & \mathbf{Set} \end{tikzcd} \tag{5}$$

▶ **Definition 14** ($\mathbf{GMet}$ equation)**.** *A* $\mathbf{GMet}$ equation *over Σ is a triple comprising a metric space (X, d) of variables, also called context, and a pair of terms $s, t \in T_\Sigma X$ that we denote by $\forall(X, d).s = t$. We say it is* satisfied *in a* $\mathbf{GMet}$ *Σ–algebra $\mathbb{A} = (A, d_A, [\![\Sigma]\!]_A)$, and we write $\mathbb{A} \vDash \forall(X, d).s = t$, if for all nonexpansive assignments $\iota : (X, d) \to (A, d_A)$, $[\![s]\!]^\iota_A = [\![t]\!]^\iota_A$.*

A $\mathbf{GMet}$ quantitative equation *over Σ is a quadruple comprising a context (X, d), a pair of terms $s, t \in T_\Sigma X$, and a bound $\varepsilon \in [0, 1]$ that we denote by $\forall(X, d).s =_\varepsilon t$. We say it is* satisfied *in a* $\mathbf{GMet}$ *Σ–algebra $\mathbb{A} = (A, d_A, [\![\Sigma]\!]_A)$, and we write $\mathbb{A} \vDash \forall(X, d).s =_\varepsilon t$, if for all nonexpansive assignments $\iota : (X, d) \to (A, d_A)$, $d_A([\![s]\!]^\iota_A, [\![t]\!]^\iota_A) \leq \varepsilon$.*

Given a class $\widehat{E}$ of $\mathbf{GMet}$ *equations and quantitative equations, we denote by $\mathbf{QAlg}(\Sigma, \widehat{E})$ the full subcategory of $\mathbf{QAlg}(\Sigma)$ of all* $\mathbf{GMet}$ *Σ–algebras that satisfy all of $\widehat{E}$.*

▶ Remark 15. In practice, we do not specify a $\mathbf{GMet}$ (quantitative) equation by giving the full description of the context. We give only distances between variables that are required, and the rest are understood to be the largest possible distances that ensure the resulting space is in $\mathbf{GMet}$. For instance, when writing $\forall x, y, z.s = t$, the context is the discrete space on $\{x, y, z\}$. In particular, any equation in the sense of Definition 8 can be interpreted as a $\mathbf{GMet}$ equation where the context is taken with the discrete metric. When writing the $\mathbf{Met}$ equation $\forall x =_\varepsilon y, y =_\delta z.s = t$, the metric space of variables is the metric d on $\{x, y, z\}$ with $d(x, y) = d(y, x) = \varepsilon$, $d(y, z) = d(z, y) = \delta$, $d(x, z) = d(z, x) = \varepsilon + \delta$ and all other distances are 0 (to ensure all axioms for $\mathbf{Met}$ are satisfied).

Quantitative Theories

▶ **Definition 16** (Quantitative algebraic theory)**.** *Given a class E of* $\mathbf{GMet}$ *(quantitative) equations over Σ, $\mathbf{QTh}(\mathbf{QAlg}(\Sigma, E))$ is the class of* $\mathbf{GMet}$ *(quantitative) equations that are satisfied in all quantitative algebras in $\mathbf{QAlg}(\Sigma, E)$. A* quantitative algebraic theory *is a class $\widehat{E}$ of* $\mathbf{GMet}$ *(quantitative) equations over a signature Σ such that $\widehat{E} = \mathbf{QTh}(\mathbf{QAlg}(\Sigma, \widehat{E}))$. For any set of* $\mathbf{GMet}$ *(quantitative) equations E, $\mathbf{QTh}(\mathbf{QAlg}(\Sigma, E))$ is a quantitative algebraic theory, and we call elements of E the* generators *of this theory.*

Without presenting a full deductive system for $\mathbf{GMet}$ (quantitative) equations, we will need the following results saying that quantitative theories are closed under some deductive rules.

▶ **Lemma 17.** *For any quantitative algebra $\mathbb{A}$, metric space (X, d), and $x, y \in X$,*

$$\mathbb{A} \vDash \forall(X, d).x =_{d(x,y)} y.$$

▶ **Lemma 18.** *For any quantitative algebra $\mathbb{A}$, metric space (X, d), and $s, t \in T_\Sigma X$,*

$$\mathbb{A} \vDash \forall(X, d).s =_1 t.$$

▶ **Lemma 19.** *For any quantitative algebra* $\mathbb{A}$*, metric space* (X,d)*, and* $s,s',t,t' \in T_\Sigma X$*,*

$$\mathbb{A} \vDash \forall(X,d).s = s' \text{ and } \mathbb{A} \vDash \forall(X,d).s =_\varepsilon t \implies \mathbb{A} \vDash \forall(X,d).s' =_\varepsilon t$$
$$\mathbb{A} \vDash \forall(X,d).t = t' \text{ and } \mathbb{A} \vDash \forall(X,d).s =_\varepsilon t \implies \mathbb{A} \vDash \forall(X,d).s =_\varepsilon t'$$

▶ **Lemma 20.** *Fix a quantitative algebra* $\mathbb{A}$ *and a set* X*, and let* $d_\perp$ *be the discrete metric on* X*. For any other metric* d *on* X*, we have*

$$\mathbb{A} \vDash \forall(X,d_\perp).s = t \implies \mathbb{A} \vDash \forall(X,d).s = t \text{ and}$$
$$\mathbb{A} \vDash \forall(X,d_\perp).s =_\varepsilon t \implies \mathbb{A} \vDash \forall(X,d).s =_\varepsilon t.$$

Proof. Any assignment $\iota : (X,d) \to (A,d_A)$ can be precomposed with $\mathrm{id}_X : (X,d_\perp) \to (X,d)$ while preserving the interpretation, i.e. $[\![s]\!]^\iota_A = [\![s]\!]^{\iota \circ \mathrm{id}_X}_A$. ◀

▶ **Lemma 21.** *Fix a quantitative algebra* $\mathbb{A}$*, and a space* (X,d)*. For any* $\varepsilon \leq \varepsilon'$

$$\mathbb{A} \vDash \forall(X,d).s =_\varepsilon t \implies \mathbb{A} \vDash \forall(X,d).s =_{\varepsilon'} t.$$

Thus, for any quantitative algebraic theory $\widehat{E}$*,* $\forall(X,d).s =_\varepsilon t \in \widehat{E}$ *implies* $\forall(X,d).s =_{\varepsilon'} t \in \widehat{E}$*.*

▶ **Lemma 22.** *For any quantitative algebra* $\mathbb{A}$*, metric spaces* (X,d) *and* (Y,Δ) *and functions* $\sigma : X \to T_\Sigma Y$*. If*

$$\forall x,x' \in X, \mathbb{A} \vDash \forall(Y,\Delta).\sigma(x) =_{d(x,x')} \sigma(x') \text{ and} \tag{6}$$
$$\mathbb{A} \vDash \forall(X,d).s =_\varepsilon t \text{ then} \tag{7}$$
$$\mathbb{A} \vDash \forall(Y,\Delta).\sigma^*(s) =_\varepsilon \sigma^*(t), \tag{8}$$

where $\sigma^*(s)$ *is the term* s *where all occurences of the variable* $x \in X$ *has been replaced by the term* $\sigma(x)$ *and similarly for* t*. Formally,* $\sigma^* = \mu^\Sigma_Y \circ T_\Sigma\sigma : T_\Sigma X \to T_\Sigma Y$*.*

Free Algebras and Monadicity

We have a quantitative analog to Proposition 10.

▶ **Proposition 23.** *For any quantitative algebraic theory* $(\Sigma,\widehat{E})$*, the forgetful functor* $U : \mathbf{QAlg}(\Sigma,\widehat{E}) \to \mathbf{GMet}$ *that forgets about the algebra structure is strictly monadic.*

Proof sketch. We give the detailed constructions of the left adjoint via free algebras. The carrier of the free $(\Sigma,\widehat{E})$–algebra on (X,d) is the metric space $\widehat{T}_{\Sigma,\widehat{E}}(X,d)$ defined as follows. The carrier is the set of terms in $T_\Sigma X$ quotiented by the equivalence relation

$$s \equiv_{\widehat{E}} t \Leftrightarrow \forall(X,d).s = t \in \widehat{E}.$$

We denote by $[s]_{\widehat{E}}$ the equivalence class of $s \in T_\Sigma X$ in $T_\Sigma X/{\equiv_{\widehat{E}}}$, and note that it also depends on d. The metric is $d_{\widehat{E}} : T_\Sigma X/{\equiv_{\widehat{E}}} \times T_\Sigma X/{\equiv_{\widehat{E}}} \to [0,1]$ defined by

$$d_{\widehat{E}}([s],[t]) \leq \varepsilon \Leftrightarrow \forall(X,d).s =_\varepsilon t.$$

Some work is need to show $\widehat{T}_{\Sigma,\widehat{E}}(X,d) := (T_\Sigma X/{\equiv_{\widehat{E}}}, d_{\widehat{E}})$ is a metric space.

The interpretation of $\mathsf{op} : n \in \Sigma$ is defined syntactically (a bit of work is needed to show this is well-defined and nonexpansive):

$$[\![\mathsf{op}]\!]([t_1]_{\widehat{E}},\ldots,[t_n]_{\widehat{E}}) = [\mathsf{op}(t_1,\ldots,t_n)]_{\widehat{E}}.$$

The universal morphism from (X,d) to U is $\eta^{\Sigma,\widehat{E}}_{(X,d)} : (X,d) \to \widehat{T}_{\Sigma,\widehat{E}}(X,d)$ sending x to $[x]_{\widehat{E}}$. After showing U uniquely creates coequalizers of U–split pairs, we obtain a monad $\widehat{T}_{\Sigma,\widehat{E}}$ with unit $\eta^{\Sigma,\widehat{E}}$ and multiplication $\mu^{\Sigma,\widehat{E}}$ such that $\mathbf{EM}(\widehat{T}_{\Sigma,\widehat{E}}) \cong \mathbf{QAlg}(\Sigma,\widehat{E})$. The explicit definitions of $\widehat{T}_{\Sigma,\widehat{E}}$ applied to $f : (A,d_A) \to (B,d_B)$ and the multiplication are respectively

$$\widehat{T}_{\Sigma,\widehat{E}}f : \widehat{T}_{\Sigma,\widehat{E}}(A,d_A) \to \widehat{T}_{\Sigma,\widehat{E}}(B,d_B) = [t]_{\widehat{E}} \mapsto [T_\Sigma f(t)]_{\widehat{E}}, \text{ and}$$

$$\mu^{\Sigma,\widehat{E}}_{(X,d)} : \widehat{T}_{\Sigma,\widehat{E}}\widehat{T}_{\Sigma,\widehat{E}}(X,d) \to \widehat{T}_{\Sigma,\widehat{E}}(X,d) = [t([t_1]_{\widehat{E}},\dots,[t_n]_{\widehat{E}})]_{\widehat{E}} \mapsto [t(t_1,\dots,t_n)]_{\widehat{E}}. \blacktriangleleft$$

Monad Presentations

▶ **Definition 24** (**Set** presentation)**.** *A* presentation *of a monad* (M,η,μ) *on* **Set** *is an algebraic theory* (Σ,E) *along with a monad isomorphism* $T_{\Sigma,E} \cong M$*.*

By Propositions 4 and 10, a presentation (Σ,E) for M yields an isomorphism of categories $\mathbf{EM}(M) \cong \mathbf{Alg}(\Sigma,E)$. If $\rho : T_{\Sigma,E} \to M$ is the isomorphism witnessing the presentation, this isomorphism of categories is given as follows.

Given an M–algebra $\alpha : MA \to A$, the algebra $\mathbb{A}_\alpha = (A, [\![\Sigma]\!]_\alpha)$ is defined by letting, for each $\mathsf{op} : n \in \Sigma$, $[\![\mathsf{op}]\!]_\alpha(a_1,\dots,a_n) = \alpha(\rho_A[\mathsf{op}(a_1,\dots,a_n)]_E)$. This interpretation extended to terms yields

$$[\![-]\!]_\alpha = T_\Sigma A \xrightarrow{[-]_E} T_{\Sigma,E}A \xrightarrow{\rho_A} MA \xrightarrow{\alpha} A.$$

Given a (Σ,E)–algebra $\mathbb{A} = (A,[\![\Sigma]\!]_A)$, the algebra $\alpha_{\mathbb{A}}$ is defined by factorising the interpretation of terms through $T_{\Sigma,E}A$ and precomposing by ρ_A^{-1}, that is,

$$\alpha_{\mathbb{A}} = MA \xrightarrow{\rho_A^{-1}} T_{\Sigma,E}A \xrightarrow{[\![-]\!]_A} A.$$

▶ **Example 25.** We give two main examples of monads on **Set** with a presentation.

1. The finite non-empty powerset monad $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$ represents nondeterminism in computation, and it is presented by the theory of semilattices comprising a binary operation $\oplus$ and the equations (stating $\oplus$ is idempotent, commutative and associative)

$$\forall x.x \oplus x = x,\ \forall x,y.x \oplus y = y \oplus x, \text{ and } \forall x,y,z.x \oplus (y \oplus z) = (x \oplus y) \oplus z. \tag{9}$$

   We will denote the signature of semilattices by Σ_{S}, the equations in (9) by E_{S}, the corresponding monad by $T_{\mathsf{S}} := T_{\Sigma_{\mathsf{S}},E_{\mathsf{S}}}$ and the isomorphism by $\rho^{\mathsf{S}} : T_{\mathsf{S}} \to \mathcal{P}$.
2. The finitely supported distributions monad $\mathcal{D} : \mathbf{Set} \to \mathbf{Set}$ represents probabilistic nondeterminism in computation, and it is presented by the theory of convex algebras comprising a binary operation $+_p$ for every $p \in (0,1)$ and the following equations (stating $+_p$ is idempotent, skew commutative and skew associative) for every $p,q \in (0,1)$

$$\forall x.x = x{+_p}x,\ \forall x,y.x{+_p}y = y{+_{1-p}}x, \text{ and } \forall x,y,z.(x{+_q}y){+_p}z = x{+_{pq}}(y{+_{\frac{p(1-q)}{1-pq}}}z). \tag{10}$$

   We will denote the signature of convex algebras by Σ_{CA}, the equations in (10) by E_{CA}, the corresponding monad by $T_{\mathsf{CA}} := T_{\Sigma_{\mathsf{CA}},E_{\mathsf{CA}}}$ and the isomorphism by $\rho^{\mathsf{CA}} : T_{\mathsf{CA}} \to \mathcal{D}$.

▶ **Definition 26** (**GMet** presentation)**.** *A presentation of a monad* $(\widehat{M},\widehat{\eta},\widehat{\mu})$ *on* **GMet** *is a quantitative algebraic theory* $(\Sigma,\widehat{E})$ *along with a monad isomorphism* $\widehat{T}_{\Sigma,\widehat{E}} \cong \widehat{M}$*.*

4 Lifting Set Presentations to GMet Presentations

Most examples of **GMet** presentations in the literature [14, 19, 17, 18] are built on top of a **Set** presentation. In summary, there is a monad M with a known algebraic presentation (Σ, E) (e.g. $\mathcal{P}$ and semilattices or $\mathcal{D}$ and convex algebras) and a lifting of every metric space (X, d) to a metric space $(MX, \widehat{d})$. Then, a quantitative algebraic theory $(\Sigma, \widehat{E})$ over the same signature is generated by counterparts to the equations in E as well as new quantitative equations to model the lifting. Finally, it is shown how the theory axiomatises the lifting, namely the **GMet** monad induced by the theory is isomorphic to a monad whose action on objects is the assignment $(X, d) \mapsto (MX, \widehat{d})$.

In this section, we prove our main result (Theorem 32) which makes this process more automatic and gives a necessary and sufficient conditions for when it can actually be done. Throughout, we fix a monad (M, η, μ) on **Set** and an algebraic theory (Σ, E) presenting M via the isomorphism $\rho : T_{\Sigma,E} \Rightarrow M$. We first give multiple definitions to make precise what we mean by *lifting*.

▶ **Definition 27.** *A* lifting *of M to* **GMet** *is an assignment $(X, d) \mapsto (MX, \widehat{d})$ defining a metric on MX for every metric on X, we denote such a lifting with $\widehat{M}$. We call it a* functor lifting *when for every nonexpansive function $f : (X, d) \to (Y, \Delta)$, $Mf : (MX, \widehat{d}) \to (MY, \widehat{\Delta})$ is also nonexpansive. This defines a functor $\widehat{M}$:* **GMet** $\to$ **GMet** *with $\widehat{M}(X, d) = (MX, \widehat{d})$ and $\widehat{M}(f) = Mf$. We call it a* monad lifting *when, it is a functor lifting and additionally, for any (X, d), the functions $\eta_X : (X, d) \to (MX, \widehat{d})$ and $\mu_X : (MMX, \widehat{\widehat{d}}) \to (MX, \widehat{d})$ are nonexpansive. This defines a monad $(\widehat{M}, \widehat{\eta}, \widehat{\mu})$ with $\widehat{M}$ being the functor defined above, $\widehat{\eta}_{(X,d)} = \eta_X$ and $\widehat{\mu}_{(X,d)} = \mu_X$.*

▶ **Definition 28.** *A* lifting *of an algebraic theory (Σ, E) to* **GMet** *is a quantitative algebraic theory $\widehat{E}$ over the same signature Σ such that for any space (X, d) and terms $s, t \in T_\Sigma X$,*

$$\forall X.s = t \in E \Leftrightarrow \forall (X, d).s = t \in \widehat{E}. \tag{11}$$

We say this lifting axiomatises *a lifting of M if for any (X, d) and terms $s, t \in T_\Sigma X$,*

$$\widehat{d}(\rho_X[s], \rho_X[t]) \leq \varepsilon \Leftrightarrow \forall (X, d).s =_\varepsilon t \in \widehat{E}. \tag{12}$$

Because any quantitative theory induces a monad (Proposition 23), the notion of theory lifting is already strong enough to induce a monad lifting.

▶ **Lemma 29.** *If $\widehat{E}$ is a lifting of a theory E, then $\widehat{T}_{\Sigma,\widehat{E}}$ is a monad lifting of $T_{\Sigma,E}$.*

Proof. By Definition 28 and the constructions of $T_{\Sigma,E}$ and $\widehat{T}_{\Sigma,\widehat{E}}$, we find that for any (X, d), $T_{\Sigma,E}X$ is the underlying set of $\widehat{T}_{\Sigma,\widehat{E}}(X, d)$. Indeed, both these sets are $T_\Sigma X$ quotiented by $\equiv_E$ and $\equiv_{\widehat{E}}$ respectively, where

$$s \equiv_E t \Leftrightarrow \forall X.s = t \in E \overset{(11)}{\Leftrightarrow} \forall (X, d).s = t \in \widehat{E} \Leftrightarrow s \equiv_{\widehat{E}} t.$$

Since the actions on morphisms, units and multiplications of both monads are defined syntactically in the same way, we conclude that $\widehat{T}_{\Sigma,\widehat{E}}$ is a monad lifting of $T_{\Sigma,E}$. ◀

If $\widehat{E}$ is a lifting of E and it axiomatises $\widehat{M}$, then we can show $\widehat{M}$ is a monad lifting by exhibiting an isomorphism $\widehat{T}_{\Sigma,\widehat{E}} \cong \widehat{M}$ that relies on the already known isomorphism $\rho : T_{\Sigma,E} \Rightarrow M$.

▶ **Lemma 30.** *If $\widehat{E}$ is a lifting of E and it axiomatises $\widehat{M}$, then $\widehat{M}$ is a monad lifting of M.*

Proof. Suppose $\widehat{E}$ is a lifting of the theory E axiomatising M. We saw in Lemma 29 that $\widehat{T}_{\Sigma,\widehat{E}}$ is a monad lifting of $T_{\Sigma,E}$. Hence, ρ_X is a bijection between the underlying sets of $\widehat{T}_{\Sigma,\widehat{E}}(X,d)$ and $\widehat{M}(X,d)$. Moreover, by previous definitions, we have

$$\widehat{d}(\rho_X[s],\rho_X[t]) \leq \varepsilon \overset{(12)}{\Leftrightarrow} \forall(X,d).s =_\varepsilon t \in \widehat{E} \Leftrightarrow d_{\widehat{E}}([s],[t]) \leq \varepsilon,$$

which implies $\rho_X : \widehat{T}_{\Sigma,\widehat{E}}(X,d) \to \widehat{M}(X,d)$ is an isometry (it preserves distances). We conclude that $\rho_X : \widehat{T}_{\Sigma,\widehat{E}}(X,d) \to \widehat{M}(X,d)$ is an isomorphism (a bijective isometry). Since ρ is a monad morphism, we have the following equations for any $f : (X,d) \to (Y,\Delta)$.

$$Mf = \widehat{M}(X,d) \xrightarrow{\rho_X^{-1}} \widehat{T}_{\Sigma,\widehat{E}}(X,d) \xrightarrow{T_{\Sigma,E}f} \widehat{T}_{\Sigma,\widehat{E}}(Y,\Delta) \xrightarrow{\rho_Y} \widehat{M}(Y,\Delta)$$

$$\eta_X = (X,d) \xrightarrow{\eta_X^{\Sigma,E}} \widehat{T}_{\Sigma,\widehat{E}}(X,d) \xrightarrow{\rho_X} \widehat{M}(X,d)$$

$$\mu_X = \widehat{M}\widehat{M}(X,d) \xrightarrow{M\rho_X^{-1}} \widehat{M}\widehat{T}_{\Sigma,\widehat{E}}(X,d) \xrightarrow{\rho^{-1}_{T_{\Sigma,E}X}} \widehat{T}_{\Sigma,\widehat{E}}\widehat{T}_{\Sigma,\widehat{E}}(X,d) \xrightarrow{\mu_X^{\Sigma,E}} \widehat{T}_{\Sigma,\widehat{E}}(X,d) \xrightarrow{\rho_X} \widehat{M}(X,d)$$

All the arrows in the composites are nonexpansive either by what we just proved (ρ_X and ρ_X^{-1} are nonexpansive for any (X,d)) or because $\widehat{T}_{\Sigma,\widehat{E}}$ is a monad lifting of $T_{\Sigma,E}$ (Lemma 29). We find that $\widehat{M}$ is a monad lifting of M. ◀

In hope to get a converse, given $\widehat{M}$ a lifting of M, we can naively attempt to define a theory $E_{\widehat{M}}$ lifting E that axiomatises it. To ensure the forward implication of (11) holds, we use Lemma 20 and add the **GMet** equation $\forall(X,d_\perp).s = t$ for each equation $\forall X.s = t$ that belongs to E. To ensure the forward implication of (12) holds, we use Lemma 21 and add the **GMet** quantitative equation $\forall(X,d).s =_\varepsilon t$ for all metric space (X,d) and terms $s,t \in T_\Sigma X$ satisfying $\widehat{d}(\rho_X[s],\rho_X[t]) = \varepsilon$. Formally, $E_{\widehat{M}} = \mathbf{QTh}(\mathbf{QAlg}(\Sigma,\widehat{E}_1 \cup \widehat{E}_2))$, where

$$\widehat{E}_1 = \{\forall(X,d_\perp).s = t \mid \forall X.s = t \in E\} \text{ and } \tag{13}$$

$$\widehat{E}_2 = \left\{\forall(X,d).s =_\varepsilon t \mid \varepsilon = \widehat{d}(\rho_X[s],\rho_X[t])\right\}. \tag{14}$$

Unfortunately, the converse implications of (11) and (12) do not always hold, but Theorem 32 says they hold exactly when $\widehat{M}$ is a monad lifting. The proof relies on one key lemma.

▶ **Lemma 31.** *Let $\widehat{M}$ be a monad lifting of M and (A,d_A) be a metric space in* **GMet**. *The lifting yields a metric $\widehat{d}_A$ on MA, and the free (Σ,E)–algebra $(MA,[\![\Sigma]\!]_{\mu_A})$ on MA is obtained by passing the free M–algebra (MA,μ_A) through the isomorphism* $\mathbf{EM}(M) \cong \mathbf{Alg}(\Sigma,E)$. *Then $(MA,[\![\Sigma]\!]_{\mu_A},\widehat{d}_A)$ is a quantitative $(\Sigma,E_{\widehat{M}})$–algebra.*

Proof. A bit of unrolling shows that for an assignment $\iota : X \to MA$, the interpretation $[\![-]\!]^\iota_{\mu_A}$ is the composite

$$T_\Sigma X \xrightarrow{T_\Sigma\iota} T_\Sigma MA \xrightarrow{[-]_E} T_{\Sigma,E}MA \xrightarrow{\rho_{MA}} MMA \xrightarrow{\mu_A} MA.$$

For later use, we apply the naturality of $[-]_E$ and ρ to rewrite the composite as

$$[\![-]\!]^\iota_{\mu_A} = T_\Sigma X \xrightarrow{[-]_E} T_{\Sigma,E}X \xrightarrow{\rho_X} MX \xrightarrow{M\iota} MMA \xrightarrow{\mu_A} MA. \tag{15}$$

We show that $\mathbb{M} = (MA,\widehat{d}_A,[\![\Sigma]\!]_{\mu_A})$ is a quantitative $(\Sigma,\widehat{E})$–algebra. First, we show it satisfies the **GMet** equations in (13). If $\forall X.s = t \in E$, then the (Σ,E)–algebra underlying

$\mathbb{M}$ satisfies $\forall X.s = t$, hence for any $\iota : (X, d_\perp) \to (MA, \widehat{d}_A)$, identifying ι with its underlying function, we have $[\![s]\!]^\iota_{\mu_A} = [\![t]\!]^\iota_{\mu_A}$, and we conclude $\mathbb{M} \vDash \forall(X, d_\perp).s = t$.

Next, we show $\mathbb{M}$ satisfies the **GMet** quantitative equations in (14). Let $\forall(X, d).s =_\varepsilon t$ with $\varepsilon = \widehat{d}(\rho_X[s], \rho_X[t])$, and let $\iota : (X, d) \to (MA, \widehat{d}_A)$ be nonexpansive. We have the following derivation

$$\begin{aligned}
\widehat{d}_A\left([\![s]\!]^\iota_{\mu_A}, [\![t]\!]^\iota_{\mu_A}\right) &= \widehat{d}_A(\mu_A(M\iota(\rho_X([s]_E))), \mu_A(M\iota(\rho_X([t]_E)))) && \text{using (15)}\\
&\leq \widehat{\widehat{d}}_A(M\iota(\rho_X([s]_E)), M\iota(\rho_X([t]_E))) && \mu_A : (MMA, \widehat{\widehat{d}}_A) \to (MA, \widehat{d}_A) \text{ nonexpansive}\\
&\leq \widehat{d}(\rho_X([s]_E), \rho_X([t]_E)) && M\iota : (MX, \widehat{d}) \to (MMA, \widehat{\widehat{d}}_A) \text{ nonexpansive}\\
&= \varepsilon
\end{aligned}$$

We conclude that $\mathbb{M} \vDash \forall(X, d).s =_\varepsilon t$ for all those **GMet** (quantitative) equations in $E_{\widehat{M}}$, and hence $\mathbb{M} \in \mathbf{QAlg}(\Sigma, E_{\widehat{M}})$. ◀

▶ **Theorem 32.** *Let $\widehat{M}$ be a lifting of M to* **GMet***, then $\widehat{M}$ is a monad lifting if and only if there exists a lifting of the theory E that axiomatises $\widehat{M}$.*

Proof. The converse direction is Lemma 30. Supposing that $\widehat{M}$ is a monad lifting of M, we will show that $E_{\widehat{M}}$ is a lifting of E axiomatising $\widehat{M}$. First, we show $E_{\widehat{M}}$ is a lifting of E, i.e. for any (X, d) and $s, t \in T_\Sigma X$,

$$\forall X.s = t \in E \Leftrightarrow \forall(X, d).s = t \in E_{\widehat{M}}.$$

($\Rightarrow$) By (13) in the definition of $E_{\widehat{M}}$, we have $\forall(X, d_\perp).s = t \in E_{\widehat{M}}$. Then, Lemma 20 implies $\forall(X, d).s = t \in E_{\widehat{M}}$.

($\Leftarrow$) Now, if $\forall(X, d).s = t \in E_{\widehat{M}}$, we saw in Lemma 31 that $\mathbb{M}_{(X,d)} = (MX, \widehat{d}, [\![\Sigma]\!]_{\mu_X})$ belongs to $\mathbf{QAlg}(\Sigma, E_{\widehat{M}})$ hence $\mathbb{M}_{(X,d)} \vDash \forall(X, d).s = t$. Taking the assignment $\eta_X : (X, d) \to \widehat{M}(X, d)$ which is nonexpansive because $\widehat{M}$ is a monad lifting, we have $[\![s]\!]^{\eta_X}_{\mu_X} = [\![t]\!]^{\eta_X}_{\mu_X}$. Using (15) and the monad law $\mu_X \circ M\eta_X = \mathrm{id}_{MX}$, we find

$$\rho_X[s]_E = [\![s]\!]^{\eta_X}_{\mu_X} = [\![t]\!]^{\eta_X}_{\mu_X} = \rho_X[t]_E.$$

Finally, since ρ_X is a bijection, we have $[s]_E = [t]_E$, i.e. $\forall X.s = t \in E$.

Next, we show that $E_{\widehat{M}}$ axiomatises $\widehat{M}$. Fix (X, d) and terms $s, t \in T_\Sigma X$, we will show

$$\widehat{d}(\rho_X[s], \rho_X[t]) \leq \varepsilon \Leftrightarrow \forall(X, d).s =_\varepsilon t \in E_{\widehat{M}}.$$

($\Rightarrow$) By definition of $E_{\widehat{M}}$, writing $\varepsilon_0 = \widehat{d}(\rho[s], \rho[t])$, we know that $\forall(X, d).s =_{\varepsilon_0} t \in E_{\widehat{M}}$. Now, if $\varepsilon_0 \leq \varepsilon$, then by Lemma 21, also $\forall(X, d).s =_\varepsilon t \in E_{\widehat{M}}$.

($\Leftarrow$) As above, Lemma 31 says that $\mathbb{M}_{(X,d)}$ satisfies $\forall(X, d).s =_\varepsilon t$. Taking the assignment $\eta_X : (X, d) \to \widehat{M}(X, d)$ which is nonexpansive because $\widehat{M}$ is a monad lifting, we have

$$\widehat{d}\,(\rho_X[s], \rho_X[t]) = \widehat{d}\left([\![s]\!]^{\eta_X}_{\mu_X}, [\![t]\!]^{\eta_X}_{\mu_X}\right) \leq \varepsilon.$$

This concludes the proof that $E_{\widehat{M}}$ is a lifting of E that axiomatises $\widehat{M}$. ◀

The forward direction of this result is new, and it says that any monad lifting of a monad on **Set** with an algebraic presentation has a quantitative algebraic presentation. This also has nice theoretical consequences, it leads to a correspondence between monad liftings and theory liftings, a step in the direction of characterising monads arising from quantitative algebraic theories.

▶ **Corollary 33.** *Denoting* $\mathsf{ML}(M)$ *the class of monad liftings of* M *and* $\mathsf{TL}(E)$ *the class of theory liftings of* E*, there is a bijection* $\mathsf{ML}(M) \cong \mathsf{TL}(E)$*.*

From the point of view of categorical algebra/logic, Corollary 33 might look incomplete. We defer to future work the task of making this result into an equivalence of categories as is common practice in the aforementioned fields.

5 Applications

We have just mentioned the significance of Theorem 32 with respect to the theoretical study of quantitative algebraic reasoning. In this section, we will show how our main theorem can also help in concrete applications of this framework.

Our primary envisioned purpose for Theorem 32 is to provide a simpler and more automatic way to use the framework of quantitative algebraic reasoning in concrete situations. The expected setting is that in the study of some computational effect (a monad) with a well understood algebraic theory (presenting the monad), there arises a need for a quantitative perspective. This is realized by defining a distance on terms of the theory that depends on a distance between variables. This assembles into what we called a lifting of the monad to **GMet**, and if it can be proven that this lifting is a monad lifting, then Theorem 32 allows to reason equationally about this distance using a quantitative algebraic theory.

Unfortunately, this theory is generated by an impractical amount of **GMet** (quantitative) equations — to implement in a model checking algorithm for instance. Nevertheless, the generating sets in (13) and (14) can be a starting point to find a more manageable set of (quantitative) equations that generates the same theory. We showcase this with four examples, they rely on the **Set** presentations in Example 25.

Powerset lifting. We define the following lifting of $\mathcal{P}$ to **Met**:

$$(X,d) \mapsto (\mathcal{P}X, \widehat{d}) \text{ where } \widehat{d}(S,S') = \begin{cases} 0 & S = S' \\ d(x,y) & S = \{x\} \text{ and } S' = \{y\} \\ 1 & \text{otherwise} \end{cases}.$$

Viewing $\mathcal{P}$ as modelling nondeterminism, this lifting says that nondeterministic processes cannot be meaningfully compared (they are put at maximum distance) unless the sets of possible outcomes are the same (distance is zero) or both processes are deterministic (distance is inherited from the distance between the only possible outcomes).

▶ **Proposition 34.** *The lifting above, we denote it by* $\widehat{\mathcal{P}}$*, is a monad lifting of* $\mathcal{P}$ *to* **Met***.*

Denoting E the **Set** theory of semilattices, Theorem 32 gives us a quantitative theory $E_{\widehat{\mathcal{P}}}$ that lifts E and axiomatises $\widehat{\mathcal{P}}$. It is generated by the **Met** (quantitative) equations

$$\widehat{E}_1 = \{\forall (X, d_\perp).s = t \mid \forall X.s = t \in E\} \text{ and } \widehat{E}_2 = \left\{\forall (X,d).s =_\varepsilon t \mid \varepsilon = \widehat{d}\left(\rho_X^{\mathsf{S}}[s], \rho_X^{\mathsf{S}}[t]\right)\right\}.$$

In order to obtain a generating set that is more convenient, we first note that since E is generated by the equations in (9), we can (using Remark 15) see them as **Met** equations that can replace $\widehat{E}_1$. We prove this in full generality.

▶ **Lemma 35.** *Let* E *and* E' *be two classes of equations over* Σ *such that for all* $\mathbb{A} \in \mathbf{Alg}(\Sigma)$*,* $\mathbb{A} \vDash E$ *implies* $\mathbb{A} \vDash E'$*. If*

$$\widehat{E} = \{\forall (X, d_\perp).s = t \mid \forall X.s = t \in E\} \quad \textit{and} \quad \widehat{E}' = \{\forall (X, d_\perp).s = t \mid \forall X.s = t \in E'\},$$

then for all $\mathbb{A} \in \mathbf{QAlg}(\Sigma)$*,* $\mathbb{A} \vDash \widehat{E}$ *implies* $\mathbb{A} \vDash \widehat{E}'$*.*

Proof. Since all assignments $\iota : X \to A$ are nonexpansive assignments $\iota : (X, d_\perp) \to (A, d_A)$ and vice versa, an algebra $\mathbb{A} \in \mathbf{QAlg}(\Sigma)$ satisfies $\forall(X, d_\perp).s = t$ if and only if its underlying algebra $U\mathbb{A} \in \mathbf{Alg}(\Sigma)$ satisfies $\forall X.s = t$. Thus, we have for all $\mathbb{A} \in \mathbf{QAlg}(\Sigma)$,

$$\mathbb{A} \vDash \widehat{E} \Leftrightarrow U\mathbb{A} \vDash E \implies U\mathbb{A} \vDash E' \Leftrightarrow \mathbb{A} \vDash \widehat{E}'. \qquad \blacktriangleleft$$

Next, we observe that all the quantitative equations in $\widehat{E}_2$ are redundant. As in the definition of $\widehat{d}$, there are three cases.

- If $[s] = [t]$, i.e. s and t represent the same subset of X, then the equation $\forall X.s = t$ is in E which means $\forall(X, d_\perp).s = t$ is in $\widehat{E}_1$. We conclude, by Lemma 20, that $\forall(X, d).s = t$ is in the theory generated by $\widehat{E}_1$ and since we are in **Met** where all self-distances are zero, it follows that $\forall(X, d).s =_0 t$ is already in the theory generated by $\widehat{E}_1$.
- If $[s] = [x]$ and $[t] = [y]$ for some $x, y \in X$, then the equations $\forall X.s = x$ and $\forall X.t = y$ are in E which means (using Lemma 20) that $\forall(X, d).s = x$ and $\forall(X, d).t = y$ are in the theory generated by $\widehat{E}_1$. Furthermore, Lemma 17 implies $\forall(X, d).x =_\varepsilon y$ is also in the theory generated by $\widehat{E}_1$ where $\varepsilon = d(x, y) = \widehat{d}(\rho_X^{\mathsf{S}}[s], \rho_X^{\mathsf{S}}[t])$, and finally by Lemma 19, $\forall(X, d).s =_\varepsilon t$ already belongs to the theory generated by $\widehat{E}_1$.
- In all other cases, $\varepsilon = \widehat{d}(\rho_X^{\mathsf{S}}[s], \rho_X^{\mathsf{S}}[t]) = 1$, so Lemma 18 implies $\forall(X, d).s =_\varepsilon t$ already belongs to the theory generated by $\widehat{E}_1$.

We conclude that $E_{\widehat{\mathcal{P}}}$ is generated by the **Met** equations

$$\forall x.x \oplus x = x,\ \forall x, y.x \oplus y = y \oplus x, \text{ and } \forall x, y, z.x \oplus (y \oplus z) = (x \oplus y) \oplus z. \tag{16}$$

In [14], the Hausdorff distance between finite subsets of a metric space is shown to be axiomatised by a quantitative algebraic theory lifting the theory of semilattices, yielding another monad lifting of $\mathcal{P}$. That theory is generated by the **Met** equations in (16) plus the set of **Met** quantitative equations below stipulating that the semilattice operation is a nonexpansive map $(A, d_A) \times (A, d_A) \to (A, d_A)$.

$$E_{\mathsf{H}} = \left\{\forall x =_\varepsilon x', y =_{\varepsilon'} y'.x \oplus y =_{\max\{\varepsilon,\varepsilon'\}} x' \oplus y' \mid \varepsilon, \varepsilon' \in [0, 1]\right\} \tag{17}$$

These quantitative equations are there by default in [14] because they only consider quantitative algebras with operations that are nonexpansive with respect to the product metric. It is then natural to ask whether the monad lifting $\widehat{\mathcal{P}}$ we defined can be presented by a quantitative algebraic theory in the sense of [14]. The answer is negative because of a property that all monads presented by theories of [14] share: they are enriched over $(\mathbf{Met}, \otimes, \mathbf{1})$ (see [2, p. 23]). The monad $\widehat{\mathcal{P}}$ is not enriched because it does not satisfy

$$\forall f, g : (X, d) \to (Y, \Delta),\ \sup_{x \in X} \Delta(f(x), g(x)) \geq \sup_{S \in \mathcal{P}X} \widehat{\Delta}(f(S), g(S)).$$

Let f be the identity function on $[0, \frac{1}{2}]$ and g be the squaring function, then the left hand side is at most $\frac{1}{2}$ (Δ is bounded by $\frac{1}{2}$), and the right hand side is 1 as witnessed by $S = \{0, \frac{1}{2}\}$: $f(S) = S$ and $g(S) = \{0, \frac{1}{4}\}$, so $\widehat{\Delta}(f(S), g(S)) = 1$.

Hausdorff lifting. The Hausdorff lifting $\widehat{\mathcal{P}}_{\mathsf{H}}$ is defined by How do we prove this is a monad lifting?

$$(X, d) \mapsto (\mathcal{P}X, d_{\mathsf{H}}) \text{ where } d_{\mathsf{H}}(S, T) = \max\left\{\max_{x \in S} \min_{y \in T} d(x, y), \max_{y \in T} \min_{x \in S} d(x, y)\right\}.$$

The proof in [14] of the axiomatisation of this lifting by $E_{\mathsf{S}} \cup E_{\mathsf{H}}$ relies on the following lemma called Hausdorff duality.

▶ **Lemma 36.** *[14, Theorem 10.5][16, Proposition 2.1] For any $S, T \in \mathcal{P}X$,*

$$d_{\mathsf{H}}(S,T) = \min\left\{\max_{(x,y)\in C} d(x,y) \mid C \subseteq X \times X, \pi_1(C) = S, \pi_2(C) = T\right\}.$$

Our general theorem cannot waive the need for this result specific to the Hausdorff lifting, but it will help streamline the axiomatisation proof by removing a lot of overhead. Using Theorem 32 and Lemma 36, it is relatively easy to show the monad $\widehat{\mathcal{P}}_{\mathsf{H}}$ is presented by the **Met** (quantitative) equations in $E_{\mathsf{S}} \cup E_{\mathsf{H}}$ (essentially Corollary 10.9 in [14]). Since $\widehat{\mathcal{P}}_{\mathsf{H}}$ is a monad lifting, Theorem 32 gives a theory $E_{\widehat{\mathcal{P}}_{\mathsf{H}}}$ presenting the monad generated by

$$\widehat{E}_1 = \{\forall (X, d_\perp).s = t \mid \forall X.s = t \in E\} \text{ and } \widehat{E}_2 = \left\{\forall (X,d).s =_\varepsilon t \mid \varepsilon = d_{\mathsf{H}}\left(\rho_X^{\mathsf{S}}[s], \rho_X^{\mathsf{S}}[t]\right)\right\}.$$

By Lemma 35, we can replace $\widehat{E}_1$ by the equations in E_{S} seen as **Met** equations. It remains to show that if a quantitative algebra $\mathbb{A} \in \mathbf{QAlg}(\Sigma_{\mathsf{S}})$ satisfies the equations in $\widehat{E}_1$ and E_{H} (we note the latter is a subset of $\widehat{E}_2$), then it also satisfies the equations in $\widehat{E}_2$. Suppose $\mathbb{A} \vDash \widehat{E}_1 \cup E_{\mathsf{H}}$, and let (X,d) be a metric space and $s,t \in T_{\mathsf{S}}X$, we will show that $\mathbb{A} \vDash s =_\varepsilon t$ with $\varepsilon = d_{\mathsf{H}}(\rho_X^{\mathsf{S}}[s], \rho_X^{\mathsf{S}}[t])$.

Lemma 36 says there exists some $C \subseteq X \times X$ satisfying $\pi_1(C) = \rho_X^{\mathsf{S}}[s]$ and $\pi_2(C) = \rho_X^{\mathsf{S}}[t]$ such that $\varepsilon = \max_{(x,y)\in C} d(x,y)$. The conditions on the projections mean that the terms $s' = \bigoplus_{c\in C} \pi_1(c)$ and $t' = \bigoplus_{c \in C} \pi_2(c)$ can be proven equal to s and t respectively in the theory of semilattices. Using Lemmas 35 and 20, we find $\mathbb{A}$ satisfies $\forall (X,d).s = s'$ and $\forall (X,d).t = t'$. Moreover, since $\mathbb{A} \vDash E_{\mathsf{H}}$, the interpretation of the semilattice operation is nonexpansive with respect to the product metric, and this implies for any assignment $\iota : (X,d) \to (A, d_A)$,

$$d_A([\![s']\!]_A^\iota, [\![t']\!]_A^\iota) \leq \max_{c\in C} d_A([\![\pi_1(c)]\!]_A^\iota, [\![\pi_2(c)]\!]_A^\iota) \leq \max_{c\in C} d(\pi_1(c), \pi_2(c)) = \varepsilon.$$

We conclude that $\mathbb{A} \vDash s' =_\varepsilon t'$ and by Lemma 19, $\mathbb{A} \vDash s =_\varepsilon t$ as desired.

Kantorovich lifting. We quickly mention a similar example for the Kantorovich lifting of $\mathcal{D}$ that was proven to be a monad lifting in [26]. After proving a convexity property of the Kantorovich metric [19, Proposition 46] and the Kantorovich–Rubinstein duality [27, Theorem 5.10], an argument close to the one above shows that the Kantorovich lifting is presented by the **Met** equations in (10) and the following set of **Met** quantitative equations.

$$E_{\mathsf{K}} = \left\{\forall x =_\varepsilon x', y =_{\varepsilon'} y'.x +_p y =_{p\varepsilon + (1-p)\varepsilon'} x' +_p y' \mid \varepsilon, \varepsilon' \in [0,1], p \in (0,1)\right\}$$

We cannot readily compare this with the presentation proof in [14] because they deal with all p-Wasserstein metrics (of which Kantorovich is an example) at once.

Hausdorff–Kantorovich lifting. In [19], the authors showed how to combine the Hausdorff lifting and the Kantorovich lifting to get a monad lifting of the monad $\mathcal{C}$ of finitely generated convex sets of distributions. They also show that the resulting monad is presented by the combination of E_{S}, E_{CA}, E_{H}, E_{K} and **Met** equations stating the distributivity of $+_p$ over $\oplus$. This presentation proof can again be streamlined using Theorem 32 and their key results.

ŁK lifting. Let us give one last example in full details. Given a metric space (X,d) and two probability distributions $\varphi, \psi \in \mathcal{D}X$, the Łukaszyk–Karmowski (ŁK for short) distance between them is

$$d_{\text{ŁK}}(\varphi,\psi) = \sum_{x,x'\in X} \varphi(x)\psi(x')d(x,x').$$

It was shown in [18] that the ŁK distance yields a monad lifting $\widehat{\mathcal{D}}$ of $\mathcal{D}$ to **DMet**, the category of diffuse metric spaces (points may have non-zero self-distance, see [8]). The authors also gave a relatively simple quantitative algebraic theory presenting it, but Theorem 32 will help us find a simpler one. Let E be the algebraic theory of convex algebras. The theorem gives us a theory $E_{ŁK}$ presenting $\widehat{\mathcal{D}}$ and generated by the **DMet** (quantitative) equations

$$\widehat{E}_1 = \{\forall(X, d_\perp).s = t \mid \forall X.s = t \in E\} \text{ and } \widehat{E}_2 = \left\{\forall(X, d).s =_\varepsilon t \mid \varepsilon = d_{ŁK}\left(\rho_X^{\mathsf{CA}}[s], \rho_X^{\mathsf{CA}}[t]\right)\right\}. \blacksquare$$

As above (using Lemma 35), we can replace $\widehat{E}_1$ by the set of equations coming from (10). In order to simplify $\widehat{E}_2$, we rely on a property that $d_{ŁK}$ satisfies: for any $\varphi, \varphi', \psi \in \mathcal{D}X$ and $p \in [0, 1]$,

$$d_{ŁK}(p\varphi + (1-p)\varphi', \psi) = pd_{ŁK}(\varphi, \psi) + (1-p)d_{ŁK}(\varphi', \psi). \tag{18}$$

Intuitively, this means that we can compute the distance between s and t by decomposing the terms into their variables, computing simple distances, then combining them to get back to s and t. Formally, we only need to keep the quantitative equations in $\widehat{E}_2$ that belong to

$$\widehat{E}'_2 = \{\forall x =_{\varepsilon_1} y, x =_{\varepsilon_2} z.x =_{p\varepsilon_1 + (1-p)\varepsilon_2} y +_p z \mid \varepsilon_1, \varepsilon_2 \in [0, 1], p \in (0, 1)\}.$$

We will prove that for any $\mathbb{A} \in \mathbf{Alg}(\Sigma_{\mathsf{CA}})$, $\mathbb{A} \vDash \widehat{E}_1 \cup \widehat{E}'_2$ implies $\mathbb{A} \vDash \widehat{E}_1 \cup \widehat{E}_2$. Suppose $\mathbb{A} \vDash \widehat{E}_1 \cup \widehat{E}'_2$, we proceed by induction on the structure of s and t to show that $\mathbb{A} \vDash \forall(X, d).s =_\varepsilon t$, where $\varepsilon = d_{ŁK}\left(\rho_X^{\mathsf{CA}}[s], \rho_X^{\mathsf{CA}}[t]\right)$. If s and t are variables, then $\rho_X^{\mathsf{CA}}[s] = \delta_x$ and $\rho_X^{\mathsf{CA}}[t] = \delta_y$ for some $x, y \in X$, thus $\varepsilon = d(x, y)$ and $\forall(X, d).x =_{d(x,y)} y$ is satisfied by $\mathbb{A}$ (by Lemma 17). Otherwise, without loss of generality (using symmetry), write $t = t_1 +_p t_2$, $\varepsilon_i = d_{ŁK}\left(\rho_X^{\mathsf{CA}}[s], \rho_X^{\mathsf{CA}}[t_i]\right)$, and the induction hypothesis tells us that $\mathbb{A} \vDash \forall(X, d).s =_{\varepsilon_i} t_i$ for $i = 1, 2$. Then, we define a substitution map $\sigma : \{x, y, z\} \to T_\Sigma X$ with $x \mapsto s$, $y \mapsto t_1$ and $z \mapsto t_2$, and since $\mathbb{A} \vDash \forall x =_{\varepsilon_1} y, x =_{\varepsilon_2} z.x =_{p\varepsilon_1+(1-p)\varepsilon_2} y +_p z \in \widehat{E}'_2$, we can apply Lemma 22 to get the desired $\mathbb{A} \vDash \forall(X, d).s =_{\varepsilon'} t$ with

$$\begin{aligned}
\varepsilon' &= pd_{ŁK}\left(\rho_X^{\mathsf{CA}}[s], \rho_X^{\mathsf{CA}}[t_1]\right) + (1-p)d_{ŁK}\left(\rho_X^{\mathsf{CA}}[s], \rho_X^{\mathsf{CA}}[t_2]\right) \\
&= d_{ŁK}\left(\rho_X^{\mathsf{CA}}[s], p\rho_X^{\mathsf{CA}}[t_1] + (1-p)\rho_X^{\mathsf{CA}}[t_2]\right) && \text{by (18)} \\
&= d_{ŁK}\left(\rho_X^{\mathsf{CA}}[s], \rho_X^{\mathsf{CA}}[t_1 +_p t_2]\right) \\
&= d_{ŁK}\left(\rho_X^{\mathsf{CA}}[s], \rho_X^{\mathsf{CA}}[t]\right) = \varepsilon.
\end{aligned}$$

We conclude that $E_{ŁK}$ is generated by the **DMet** equations in (10) and the **DMet** quantitative equations in $\widehat{E}'_2$.

6 Conclusion and Future Work

We have presented an automatic process for constructing a quantitative algebraic presentation of a monad lifting on **GMet**, given a algebraic presentation for its underlying monad on **Set** with an algebraic presentation. While this presentation may not be practically convenient, we have shown how it can guide the search for simpler presentations.

As we continue to work towards growing the list of presentation results, we believe that our approach can be useful in several ways. For instance, verifying that the multiplication of a monad is nonexpansive (for some lifting) can be very difficult. Thus, it could lessen the burden if we can find a property of the quantitative theory given in (13) and (14) that is equivalent to the multiplication being nonexpansive. Additionally, understanding these theories might help in determining when two monad liftings can be composed, given that there is a (weak) distributive law between the underlying monads. This is not always the case [17, Theorem 44].

In Corollary 33, we hint at a small step towards a correspondence between monads on **GMet** and quantitative algebraic theories. More work is needed, especially after noting that the results of [1] and [2] do not apply to our framework as they work with enriched monads.

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7 Appendix

7.1 Proofs of Section 3

7.1.1 Proof of Lemma 21

Suppose $\mathbb{A} \vDash \forall(X,d).s =_\varepsilon t$. For any $\iota : (X,d) \to (A, d_A)$, we have $d_A(\llbracket s \rrbracket^\iota_A, \llbracket t \rrbracket^\iota_A) \leq \varepsilon \leq \varepsilon'$, so $\mathbb{A} \vDash \forall(X,d).s =_{\varepsilon'} t$. It follows that

$$\begin{aligned}
\forall(X,d).s =_\varepsilon t \in \widehat{E} &\Leftrightarrow \forall \mathbb{A} \in \mathbf{QAlg}(\Sigma, \widehat{E}), \mathbb{A} \vDash \forall(X,d).s =_\varepsilon t \\
&\implies \forall \mathbb{A} \in \mathbf{QAlg}(\Sigma, \widehat{E}), \mathbb{A} \vDash \forall(X,d).s =_{\varepsilon'} t \\
&\Leftrightarrow \forall(X,d).s =_{\varepsilon'} t \in \widehat{E}.
\end{aligned}$$

7.1.2 Proof of Lemma 22

Suppose (6) and (7) hold and let $\iota : (Y, \Delta) \to (A, d_A)$. Define the assignment $\iota_\sigma : (X,d) \to (A, d_A)$ that sends $x \in X$ to $\llbracket \sigma(x) \rrbracket^\iota_A \in A$. It is nonexpansive because for any $x, x' \in X$, $d_A(\llbracket \sigma(x) \rrbracket^\iota_A, \llbracket \sigma(x') \rrbracket^\iota_A) \leq d(x,x')$ by (6). Therefore, by (7), $d_A(\llbracket s \rrbracket^{\iota_\sigma}_A, \llbracket t \rrbracket^{\iota_\sigma}_A) \leq \varepsilon$. Finally, we observe that

$$\begin{aligned}
\llbracket - \rrbracket^{\iota_\sigma}_A &= \llbracket - \rrbracket_A \circ T_\Sigma(\iota_\sigma) \\
&= \llbracket - \rrbracket_A \circ T_\Sigma(\llbracket \sigma(-) \rrbracket^\iota_A) \\
&= \llbracket - \rrbracket_A \circ T_\Sigma \left(\llbracket - \rrbracket_A \circ T_\Sigma \iota \circ \sigma\right) \\
&= \llbracket - \rrbracket_A \circ T_\Sigma \llbracket - \rrbracket_A \circ T_\Sigma T_\Sigma \iota \circ T_\Sigma \sigma \\
&= \llbracket - \rrbracket_A \circ \mu^\Sigma_A \circ T_\Sigma T_\Sigma \iota \circ T_\Sigma \sigma \\
&= \llbracket - \rrbracket_A \circ T_\Sigma \iota \circ \mu^\Sigma_Y \circ T_\Sigma \sigma \\
&= \llbracket \sigma^*(-) \rrbracket^\iota_A,
\end{aligned}$$

so $d_A(\llbracket \sigma^*(s) \rrbracket^\iota_A, \llbracket \sigma^*(t) \rrbracket^\iota_A) \leq \varepsilon$.

7.2 Proof of Proposition 23

We prove here that the algebra constructed in the proof sketch is the free algebra, hence giving a left adjoint to the forgetful functor.

Fix a metric space (X,d) and denote $\mathbb{T}_{X,d}$ the free algebra on it. The carrier is $(T_\Sigma X/\equiv_{\widehat{E}}, d_{\widehat{E}})$ and the interpretation of operations is the syntactic one that ensures $\mathbb{T}_{(X,d)}$ belongs to $\mathbf{QAlg}(\Sigma, \widehat{E})$. For any algebra $\mathbb{A} = (A, d_A, \llbracket \Sigma \rrbracket_A)$ and nonexpansive function $f : (X,d) \to (A, d_A)$, we need to find a homomorphism $f^* : \mathbb{T}_{(X,d)} \to \mathbb{A}$ such that $f^*[x]_{\widehat{E}} = f(x)$.

Since $T_\Sigma X$ is the free Σ–algebra on X, there is a homomorphism $f^\star$ from $T_\Sigma X$ to the underlying Σ–algebra of $\mathbb{A}$ that satisfies $f^\star(x) = f(x)$. This equation and the homomorphism property imply that for any $t \in T_\Sigma X$, $f^\star(t) = \llbracket t \rrbracket^f_A$. Thus, if $[s]_{\widehat{E}} = [t]_{\widehat{E}}$ then by definition $\forall(X,d).s = t \in \widehat{E}$ which means

$$f^\star(s) = \llbracket s \rrbracket^f_A = \llbracket t \rrbracket^f_A = f^\star(t)$$

because $\mathbb{A}$ satisfies all equations in $\widehat{E}$. Factoring $f^\star$ through $T_\Sigma X/\equiv_{\widehat{E}}$, we get a well-defined homomorphism f^* between the underlying Σ–algebras of $\mathbb{T}_{(X,d)}$ and $\mathbb{A}$, and it satisfies $f^*([x]_{\widehat{E}}) = f^\star(x) = f(x)$.

It remains to show f^* is nonexpansive. Let $s,t \in T_\Sigma X$ such that $d_{\widehat{E}}([s]_{\widehat{E}}, [t]_{\widehat{E}}) = \varepsilon$, this means $\forall(X,d).s =_\varepsilon t \in \widehat{E}$. Since $\mathbb{A}$ satisfies all quantitative equations in $\widehat{E}$, we have

$$d_A(f^*[s]_{\widehat{E}}, f^*[t]_{\widehat{E}}) = d_A([\![s]\!]_A^f, [\![t]\!]_A^f) \leq \varepsilon,$$

hence f^* is nonexpansive.

The uniqueness of f^* follows from the uniqueness of $f^\star$. Indeed, let $f^\sharp$ be a homomorphism $\mathbb{T}_{(X,d)} \to \mathbb{A}$ satisfying $f^\sharp[x]_{\widehat{E}} = f(x)$, then precompose $f^\sharp$ with the quotient $T_\Sigma X \twoheadrightarrow T_\Sigma X/{\equiv_{\widehat{E}}}$. The result is a homomorphism of Σ–algebras $q : T_\Sigma X \to A$ that sends x to $f(x)$, so it is $f^\star$ by uniqueness. Now, we have $f^* \circ q = f^\star = f^\sharp \circ q$, which means $f^* = f^\sharp$ since q is surjective.

7.3 Proofs of Section 4

7.3.1 Proof of Corollary 33

Given $\widehat{M}$ a monad lifting of M, Theorem 32 showed $E_{\widehat{M}}$ is a lifting of E axiomatising $\widehat{M}$. It is a formal consequence of the definitions that if $\widehat{E}$ and $\widehat{E}'$ both lift E and axiomatize $\widehat{M}$, then $\widehat{E} = \widehat{E}'$. Indeed, for any (X,d), terms $s,t \in T_\Sigma X$ and $\varepsilon \in [0,1]$, we have

$$\forall(X,d).s = t \in \widehat{E} \overset{(11)}{\Leftrightarrow} \forall X.s = t \in E \overset{(11)}{\Leftrightarrow} \forall(X,d).s = t \in \widehat{E}', \text{ and}$$
$$\forall(X,d).s =_\varepsilon t \in \widehat{E} \overset{(12)}{\Leftrightarrow} \widehat{d}(\rho[s],\rho[t]) \leq \varepsilon \overset{(12)}{\Leftrightarrow} \forall(X,d).s =_\varepsilon t \in \widehat{E}'.$$

If $\widehat{E}$ is a lifting of E, then Lemma 29 says that $\widehat{T}_{\Sigma,\widehat{E}}$ is a monad lifting of $T_{\Sigma,E}$, and using the isomorphism $M \cong T_{\Sigma,E}$, we can define a monad lifting $M_{\widehat{E}}$ of M axiomatised by $\widehat{E}$. For any $m,m' \in MX$, let

$$\widehat{d}(m,m') = \inf\left\{\varepsilon \in [0,1] \mid \forall(X,d).s =_\varepsilon t \in \widehat{E}, \rho_X^{-1}(m) = [s] \text{ and } \rho_X^{-1}(m') = [t]\right\}.$$

It follows from Lemma 21 and Definition **??** that $\widehat{E}$ axiomatises $\widehat{M}$.

It is a formal consequence of the definitions that if $\widehat{E}$ axiomatises both $\widehat{M}$ and $\widehat{M}'$ monad liftings of M, then $\widehat{M} = \widehat{M}'$. Indeed, denoting $\widehat{M}(X,d) = (MX,\widehat{d})$ and $\widehat{M}'(X,d) = (MX,\widehat{d}')$, we have for any $m,m' \in MX$ and terms $s,t \in T_\Sigma X$ satisfying $\rho_X^{-1}(m) = [s]$ and $\rho_X^{-1}(m') = [t]$:

$$\widehat{d}(m,m') \leq \varepsilon \Leftrightarrow \widehat{d}(\rho[s],\rho[t]) \leq \varepsilon \overset{(12)}{\Leftrightarrow} \forall(X,d).s =_\varepsilon t \in \widehat{E} \overset{(12)}{\Leftrightarrow} \widehat{d}'(\rho[s],\rho[t]) \leq \varepsilon \Leftrightarrow \widehat{d}'(m,m') \leq \varepsilon. \blacksquare$$

We find that sending $\widehat{M}$ to $E_{\widehat{M}}$ and sending $\widehat{E}$ to $M_{\widehat{E}}$ are inverses, yielding the bijection $\mathsf{ML}(M) \cong \mathsf{TL}(E)$.

7.4 Proofs of Section 5

7.4.1 Proof of Proposition 34

The fact that $\widehat{\mathcal{P}}$ is a monad lifting of $\mathcal{P}$ to **Met** is a consequence of the following lemmas.

▶ **Lemma 37.** *If (X,d) is a metric space, then so is $(\mathcal{P}X,\widehat{d})$.*

Proof. Symmetry (1) is clear from the definition (using symmetry of d). We can prove (2) and (3) at once by

$$\begin{aligned}\widehat{d}(S,S') = 0 &\Leftrightarrow S = S' \text{ or } S = \{x\},\ S' = \{y\},\ d(x,y) = 0\\ &\Leftrightarrow S = S' \text{ or } S = \{x\},\ S' = \{y\},\ x = y\\ &\Leftrightarrow S = S'.\end{aligned}$$

For (4), let $S, T, U \in \mathcal{P}X$. If $\widehat{d}(S,T) = 0$, then $\widehat{d}(S,U) = \widehat{d}(T,U) = \widehat{d}(S,T) + \widehat{d}(T,U)$, and a symmetric argument works when $\widehat{d}(T,U) = 0$. If one of $\widehat{d}(S,T)$ or $\widehat{d}(T,U)$ is equal to 1, then since $\widehat{d}(S,U) \leq 1$, the triangle inequality must hold. In the last possible cases, all sets must be singletons, so

$$\widehat{d}(\{x\},\{z\}) = d(x,z) \leq d(x,y) + d(y,z) = \widehat{d}(\{x\},\{y\}) + \widehat{d}(\{y\},\{z\}).$$

◀

▶ **Lemma 38.** *If $f : (X,d) \to (Y,\Delta)$ is nonexpansive, then so is $\mathcal{P}f : (\mathcal{P}X, \widehat{d}) \to (\mathcal{P}Y, \widehat{\Delta})$.*

Proof. Let $S, S' \in \mathcal{P}X$. If $S = S'$, then $f(S) = f(S')$, so

$$\widehat{\Delta}(f(S), f(S')) = 0 \leq 0 = \widehat{d}(S,S').$$

If $S = \{x\}$ and $S' = \{y\}$, then $f(S) = \{f(x)\}$ and $f(S') = \{f(y)\}$, so

$$\widehat{\Delta}(f(S), f(S')) = \Delta(f(x), f(y)) \leq d(x,y) = \widehat{d}(S,S').$$

Otherwise, $\widehat{d}(S,S') = 1$ and $\widehat{\Delta}(f(S), f(S'))$ is always less or equal to 1. ◀

▶ **Lemma 39.** *For any (X,d), the map $\eta_X : (X,d) \to (\mathcal{P}X, \widehat{d})$ is nonexpansive.*

Proof. Recall that $\eta_X(x) = \{x\}$. For any $x, y \in X$, $\widehat{d}(\{x\},\{y\}) = d(x,y)$, so η_X is even an isometry. ◀

▶ **Lemma 40.** *For any (X,d), the map $\mu_X : (\mathcal{P}\mathcal{P}X, \widehat{\widehat{d}}) \to (\mathcal{P}X, \widehat{d})$ is nonexpansive.*

Proof. Recall that $\mu_X(\mathcal{F}) = \cup\mathcal{F}$ and let $\mathcal{F}, \mathcal{F}' \in \mathcal{P}\mathcal{P}X$. The case $\mathcal{F} = \mathcal{F}'$ is dealt with by (2) and (3). If $\mathcal{F} = \{S\}$ and $\mathcal{F}' = \{S'\}$, then

$$\widehat{d}(\mu_X(\mathcal{F}), \mu_X(\mathcal{F}') = \widehat{d}(S,S') = \widehat{\widehat{d}}(\{S\},\{S'\}).$$

In the last possible cases, $\widehat{\widehat{d}}(\mathcal{F},\mathcal{F}') = 1$, so the inequality holds because $\widehat{d}(\mu_X(\mathcal{F}), \mu_X(\mathcal{F}')$ is always less or equal to 1. ◀